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# Model Comparison Games for Horn Description Logics

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**Abstract**—Horn description logics are syntactically defined fragments of standard description logics that fall within the Horn fragment of first-order logic and for which ontology-mediated query answering is in PTIME for data complexity. They were independently introduced in modal logic to capture the intersection of Horn first-order logic with modal logic. In this paper, we introduce model comparison games for the basic Horn description logic *hornALC* (corresponding to the basic Horn modal logic) and use them to obtain an Ehrenfeucht-Fraïssé type definability result and a van Benthem style expressive completeness result for *hornALC*. We also establish a finite model theory version of the latter. The Ehrenfeucht-Fraïssé type definability result is used to show that checking *hornALC* indistinguishability of models is EXPTIME-complete, which is in sharp contrast to *ALC* indistinguishability (i.e., bisimulation equivalence) checkable in PTIME. In addition, we explore the behavior of Horn fragments of more expressive description and modal logics by defining a Horn guarded fragment of first-order logic and introducing model comparison games for it.

## I. INTRODUCTION

Description logics (DLs) [1], [2] have been introduced as knowledge representation formalisms supported by efficient reasoning tools. The basic DL, called *ALC*, is a notational variant of the classical multi-modal logic. In fact, numerous applications have led to the development of a large family of DLs with different features. DLs serve as the logical underpinning of ontologies, finitely axiomatized theories known as TBoxes. Two main reasoning problems have to be solved efficiently for TBoxes, often containing thousands of axioms:

*Deduction*: does a formula follow from a TBox?

*Query answering*: is a tuple  $\mathbf{d}$  in a database  $D$  a certain answer to an ontology-mediated query  $(\mathcal{T}, q(\mathbf{x}))$  comprising a TBox  $\mathcal{T}$  and a query  $q(\mathbf{x})$ ? In other words, does  $q(\mathbf{d})$  follow from  $\mathcal{T} \cup D$ ?

As the data is typically much larger than both TBox and query, the appropriate efficiency measure for ontology-mediated query answering is *data complexity*, under which the database is the only input to the problem, while the TBox and query are regarded as fixed [3]. For *ALC*, deduction is EXPTIME-complete and ontology-mediated query answering with conjunctive queries is CONP-complete [2].

Horn DLs have been introduced as syntactically defined fragments of standard DLs that fall within the Horn fragment

of first-order logic (henceforth Horn FO) and for which ontology-mediated query answering is in PTIME for data complexity [4], [5]. The Horn fragment of *ALC* is called *hornALC*. The modal logic corresponding to *hornALC* was introduced independently, actually five years earlier, with the aim of capturing the intersection of Horn FO and modal logic [6].<sup>1</sup> The introduction of Horn DLs had an enormous impact on description logic research and applications: while the weaker Horn DLs of the  $\mathcal{EL}$  [7] and *DL-Lite* [8], [9] families gave rise to two Web Ontology Language OWL 2 profiles (trading expressiveness for high efficiency), the more expressive Horn DLs starting at *hornALC* have also been used extensively, and investigated in depth for ontology-mediated query answering [10]–[15]. Moreover, despite the fact that deduction in many expressive Horn DLs, including *hornALC*, is EXPTIME-hard, it turned out that efficient reasoners capable of coping with very large real-world TBoxes could be developed [16], [17]. The complexity of reasoning in various types of Horn DLs has been investigated in [18]. It also turned out that basic questions relevant for ontology-mediated query answering, such as query-inseparability and conservative extensions, query emptiness, and query by example, admit more elegant solutions and are easier to solve computationally for Horn DLs than in the classical case [19]–[22]. The relationship between Horn DLs and PTIME query answering is by now well-understood [23], [24].

In contrast, the model theory for expressive Horn DLs remains largely undeveloped. Even very basic questions such as whether *hornALC* indeed captures the intersection of *ALC* (or modal logic) and Horn FO are still unanswered. The aim of this paper is to lay foundations for a model-theoretic understanding of Horn DLs (and Horn modal logic) by developing model comparison games and using them to obtain Ehrenfeucht-Fraïssé type definability and van Benthem style expressive completeness results. In a first application of these results, we show that concept learning and model indistinguishability in *hornALC* are EXPTIME-complete and that *hornALC* does not capture the intersection of *ALC* and Horn FO.

<sup>1</sup>The results obtained in this paper could have been presented as a contribution to modal rather than description logic. The only reason why we have chosen the DL environment is that the impact of Horn fragments in description logic has so far been much more significant than in modal logic.

The original definition of Horn DLs [4] was based on the polarity of concepts, as used in automated theorem proving. The equivalent definition given for modal logic [6] (and also for DLs [23]) is more similar to the classical definition of Horn FO as the closure of Horn clauses  $\varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \varphi_{n+1}$ , with atomic  $\varphi_i$ , under  $\wedge$ ,  $\exists$ , and  $\forall$ . The obvious analogue of this definition in description (modal) logic is the closure of Horn clauses  $A_1 \sqcap \dots \sqcap A_n \rightarrow A_{n+1}$ , with concept names (propositional variables in modal logic)  $A_i$ , under  $\sqcap$ ,  $\exists R$ ,  $\forall R$  (respectively,  $\wedge$ ,  $\Diamond$ ,  $\Box$ ). However, in contrast to the first-order case, this definition leaves out the substitution instances  $C_1 \sqcap \dots \sqcap C_n \rightarrow C$  with positive existential  $C_i$  and Horn  $C$  (understood recursively), which have thus been explicitly included in *hornALC*. It is easy to show that *hornALC* is a fragment of Horn FO under the standard translation.

As the first contribution of the paper, we introduce model comparison games for characterizing *hornALC*. These *Horn simulation games* differ from standard bisimulation or Ehrenfeucht-Fraïssé games in the following respects:

- (1) the Horn simulation relations underlying Horn simulation games are non-symmetric (which reflects that Horn languages are not closed under negation);
- (2) positions in the games consist of pairs  $(X, b)$  with a set  $X$  of nodes and a node  $b$  (which reflects that Horn languages are not closed under disjunction);
- (3) Horn simulation games use as a subgame the basic simulation game for checking indistinguishability by positive existential *ALC* formulas (which reflects that the left-hand side of *hornALC* implications are such).

Both (2) and (3) have important consequences. The latter means that Horn simulation games are *modular* as far as the characterization of the left-hand side of implications is concerned. For example, by dropping the subgames entirely, we characterize the positive fragment of *ALC* and by restricting it to depth 0, we characterize the weaker Horn fragment of *ALC* discussed above. We will use this modularity to characterize a proper extension, *hornALC*<sub>∇</sub>, of *hornALC* with the operators  $\nabla R.C = \exists R. \top \sqcap \forall R.C$  (or  $\nabla p = \Diamond \top \wedge \Box p$  in modal logic) on the left-hand side of *hornALC* implications, which also lies in Horn FO.

The consequences of (2) are three-fold. First, using sets rather than nodes in positions implies that the obvious algorithm checking the existence of Horn simulations containing a pair  $(\{a\}, b)$  of nodes runs in exponential time. Thus, using Horn simulation games to check whether two nodes  $a$  and  $b$  satisfy the same *hornALC*-concepts or whether two models satisfy the same TBox axioms yields exponential time algorithms. We show that this is unavoidable by proving corresponding EXPTIME lower bounds. The EXPTIME-completeness results are in sharp contrast to the typical complexity of indistinguishability in modal-like languages. For example, as bisimilarity of nodes can be checked in polynomial time, deciding whether two nodes satisfy the same *ALC*-concepts is in PTIME; similarly, since one can check in polynomial time whether there is a standard simulation be-

tween two nodes, deciding whether they satisfy the same *EL*-concepts is in PTIME as well. Thus, *hornALC* sits between languages for which definability questions are computationally and model-theoretically much more straightforward.

Second, as player 2 does *not* have a winning strategy in position  $(X, b)$  in the Horn simulation game if, and only if, there exists a *hornALC*-concept that is true at all nodes in  $X$  but not true at  $b$ , our complexity results are directly applicable to the *concept learning by example (CBE) problem*: given a data set, and sets  $P$  and  $N$  of positive and negative examples, does there exist a *hornALC*-concept  $C$  separating  $P$  from  $N$  over the data? The goal of this supervised learning problem is to automatically derive new concept descriptions from labelled data. It has been investigated before in DL [25]–[27] and for many logical languages, in particular in databases [28]–[31]. Horn DLs are of particular interest as target languages for CBE as they can be regarded as ‘maximal DLs without disjunction,’ and the unlimited use of disjunction in derived concept descriptions is undesirable as it leads to *overfitting*: learnt concepts enumerate the positive examples rather than generalize from the examples [26]. The complexity analysis for Horn simulation games shows that the CBE problem for *hornALC* is EXPTIME-complete, again in contrast to *ALC*, where CBE is in PTIME. We regard the increased complexity as the price for obtaining proper generalizations.

Finally, the presence of sets in positions of the Horn simulation games has an impact on the standard infinitary saturated model approach to proving van Benthem style expressive completeness results [32], [33]. For example, we aim to prove that an FO-formula with one free variable is equivalent to (the standard translation of) a *hornALC*-concept just in case it is preserved under Horn simulations. Then, for the infinitary proof method, not only do the structures showing that non-equivalence to a *hornALC*-concept implies non-preservation under Horn simulations have to satisfy appropriate saturatedness conditions, but also the *substructures* induced by the sets  $X$  chosen by the players have to be saturated. However, saturated structures do not enjoy the latter property for arbitrary subsets  $X$  of their domain. In fact, it currently seems that the only way to obtain expressive completeness results with an infinitary approach is to restrict the moves of players to ‘saturated sets,’ say sets definable as the intersection of FO-definable sets.

In this paper, we prove van Benthem style expressive completeness results for *hornALC*-concepts and TBoxes via Horn simulation games by developing appropriate finitary methods which do not require saturated structures. As a consequence, the results hold both in the classical and the finite model theory setting, and without any restrictions on the moves of players. In fact, we show that preservation under  $\ell$ -round Horn simulation games coincides with preservation under infinitary Horn simulation games for *ALC*-concepts and TBoxes of nesting depth  $\ell$ . We thus also obtain decidability of the problem whether an *ALC*-concept or TBox is equivalent to a *hornALC*-concept or TBox, respectively. The finitary approach to van Benthem style expressive completeness results was first used

by Rosen [34] to obtain a bisimulation characterization of  $\mathcal{ALC}$  in the finite model theory setting, and has been further developed and applied with great success by Otto et al. [33], [35]–[37]. The lifting of our results from expressive completeness for  $\text{horn}\mathcal{ALC}$  within  $\mathcal{ALC}$  to expressive completeness for  $\text{horn}\mathcal{ALC}$  within FO relies on these earlier results.

It is straightforward to extend the Horn simulation games for  $\text{horn}\mathcal{ALC}$  to games providing Ehrenfeucht-Fraïssé type definability results for the Horn fragments of many popular extensions of  $\mathcal{ALC}$ , such as the extension by inverse roles or the universal role. Instead of going through those extensions step-by-step, however, we consider the guarded fragment, GF, of FO and introduce its Horn fragment,  $\text{hornGF}$ , by generalizing the definition of Horn DLs in the obvious way. We remind the reader that GF has been introduced as an extension of multi-modal logic to predicates of arbitrary arity, which still has many of the fundamental properties of modal and description logics [32], [38]–[41]. Like  $\text{horn}\mathcal{ALC}$ ,  $\text{hornGF}$  is contained in Horn FO and ontology-mediated query answering using conjunctive queries is in PTIME for data complexity. The latter can be shown by establishing a close relationship between  $\text{hornGF}$  and guarded tuple-generating dependencies (guarded tgds), a popular member of the Datalog<sup>±</sup> family for which query answering is in PTIME [42], [43]. In fact, guarded tgds can be seen as normal forms for  $\text{hornGF}$ , and deduction and query answering in  $\text{hornGF}$  can both be polynomially reduced to the same problem for guarded tgds, and vice versa. To study the model theory of  $\text{hornGF}$ , we generalize Horn simulations to guarded Horn simulations, and show an Ehrenfeucht-Fraïssé type definability result for  $\text{hornGF}$ . This result is used to prove an EXPTIME upper bound for model indistinguishability in  $\text{hornGF}$  and to explore the expressive power of  $\text{hornGF}$ . In particular, we show that  $\text{hornGF}$  captures more of the intersection of  $\mathcal{ALC}$  and Horn FO than  $\text{horn}\mathcal{ALC}$  but does not capture the intersection of GF and Horn FO. We then show expressive completeness of  $\text{hornGF}$ : an FO-formula is equivalent to a  $\text{hornGF}$ -formula just in case it is preserved under guarded Horn simulations. Our proof uses infinitary methods and thus the moves of player 1 are restricted to intersections of FO-definable sets. It remains open whether the expressive completeness holds without this restriction and whether it holds in the finite model theory setting.

The emerging landscape of the fragments of Horn FO and GF we considered in this paper is discussed in the conclusion.

Some of the proofs are deferred to the full version of the paper which is available at <http://arxiv.org/abs/1904.06919>.

**Related Work.** Here we briefly review the related work not yet discussed. The definition of Horn simulations is inspired by games used to provide van Benthem style characterizations of concepts in weak DLs such as  $\mathcal{FL}^-$  [44]. Van Benthem style characterizations of DLs in the  $\mathcal{EL}$  and  $\text{DL-Lite}$  families are given in [45]. Bisimulations have been studied for the guarded fragment and many variations [33], [38], [46]. Bisimulations have also been studied recently for coalgebraic modal logics [47], [48] and fuzzy modal logics [49].

This paper contributes to the model theory of languages obtained by taking the intersection of Horn FO with modal and description logic. Horn FO was originally introduced in classical model theory [50], [51] with the aim of understanding FO-formulas that are preserved under products of models. As it turned out, an FO-formula is equivalent to a Horn formula iff it is preserved under *reduced* products; for details consult [52], [53]. A complicated recursive characterization of FO-sentences preserved under direct products is given in [54].

There have been other attempts to define Horn modal and temporal logics [55]–[58] with a focus on the complexity of reasoning and not on model theory.

## II. PRELIMINARIES

Description logics (DLs) are fragments of first-order logic with unary and binary predicates. However, the standard notation for DL ‘formulas’ is more succinct and does not use individual variables explicitly [1], [2]. Let  $\tau$  be a vocabulary consisting of unary and binary predicate names only. In DL parlance, they are called *concept names* (denoted  $A, B$ , etc.) and *role names* (denoted  $R, S$ , etc.), respectively. The  $\mathcal{ALC}[\tau]$ -concepts,  $C$ , are defined by the following grammar:

$$C, D ::= A \mid \top \mid \perp \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid \\ C \rightarrow D \mid \exists R.C \mid \forall R.C,$$

where  $A \in \tau$  is unary,  $R \in \tau$  binary,  $\top$  is the universal and  $\perp$  the empty concept. If not relevant, we drop  $\tau$  and simply talk about  $\mathcal{ALC}$ -concepts. An  $\mathcal{ALC}[\tau]$ -concept inclusion (or CI) takes the form  $C \sqsubseteq D$ , where  $C$  and  $D$  are  $\mathcal{ALC}[\tau]$ -concepts. An  $\mathcal{ALC}[\tau]$ -TBox,  $\mathcal{T}$ , is a finite set of  $\mathcal{ALC}[\tau]$ -CIs.

$\mathcal{ALC}[\tau]$  is interpreted in usual  $\tau$ -structures

$$\mathfrak{A} = (\text{dom}(\mathfrak{A}), (A^{\mathfrak{A}})_{A \in \tau}, (R^{\mathfrak{A}})_{R \in \tau})$$

with  $\text{dom}(\mathfrak{A}) \neq \emptyset$ ,  $A^{\mathfrak{A}} \subseteq \text{dom}(\mathfrak{A})$  and  $R^{\mathfrak{A}} \subseteq \text{dom}(\mathfrak{A})^2$ . The semantics of  $\mathcal{ALC}$  can be defined via the *standard translation*  $^{\dagger}$  of  $\mathcal{ALC}$ -concepts to FO-formulas with one free variable  $x$ :

$$A^{\dagger} = A(x), \quad \top^{\dagger} = (x = x), \quad \perp^{\dagger} = \neg(x = x),$$

$^{\dagger}$  commutes with the Booleans (changing  $\sqcap$  to  $\wedge$  and  $\sqcup$  to  $\vee$ ),

$$(\exists R.C)^{\dagger} = \exists y (R(x, y) \wedge C^{\dagger}[y/x]),$$

$$(\forall R.C)^{\dagger} = \forall y (R(x, y) \rightarrow C^{\dagger}[y/x]).$$

The *extension*  $C^{\mathfrak{A}}$  of a concept  $C$  in a structure  $\mathfrak{A}$  is defined as

$$C^{\mathfrak{A}} = \{a \in \text{dom}(\mathfrak{A}) \mid \mathfrak{A} \models C^{\dagger}(a)\},$$

and the CI  $C \sqsubseteq D$  is regarded as a shorthand for the FO-sentence  $\forall x (C^{\dagger}(x) \rightarrow D^{\dagger}(x))$ . We write  $\mathcal{T} \models C \sqsubseteq D$  to say that the CI  $C \sqsubseteq D$  follows from the TBox  $\mathcal{T}$ , that is,  $C^{\mathfrak{A}} \subseteq D^{\mathfrak{A}}$  holds in every model  $\mathfrak{A}$  of  $\mathcal{T}$ . Concepts  $C$  and  $D$  are *equivalent* if  $\emptyset \models C \sqsubseteq D$  and  $\emptyset \models D \sqsubseteq C$ .

$\mathcal{ALC}$ -concepts that are built from concept names using  $\top$ ,  $\sqcap$ ,  $\sqcup$ , and  $\exists R.C$  only are called  $\mathcal{ELU}$ -concepts;  $\mathcal{ELU}$ -concepts without  $\sqcup$  are called  $\mathcal{EL}$ -concepts. The FO-translation  $C^{\dagger}$  of any  $\mathcal{ELU}$ -concept  $C$  is clearly a positive existential formula.

**Definition 1 (Horn  $\mathcal{ALC}$ -concept).** We define  $\text{horn}\mathcal{ALC}[\tau]$ -concepts,  $H$ , by the grammar

$$H, H' ::= \perp \mid \top \mid A \mid H \sqcap H' \mid L \rightarrow H \mid \exists R.H \mid \forall R.H,$$

where  $L$  is an  $\mathcal{ELU}[\tau]$ -concept. A  $\text{horn}\mathcal{ALC}$ -CI takes the form  $L \sqsubseteq H$ . A  $\text{horn}\mathcal{ALC}$ -TBox is a finite set of  $\text{horn}\mathcal{ALC}$ -CIs.

Our definition of  $\text{horn}\mathcal{ALC}$ -concepts is from [23]. We show in the full version that both polarity-based definition of  $\text{horn}\mathcal{ALC}$ -concepts from [4] and Horn *modal* formulas (appropriately adapted to the DL syntax) defined in [6] are equivalent to our definition.

To put  $\text{horn}\mathcal{ALC}$ -concepts into the context of classical Horn FO-formulas, we recall that a *basic Horn formula* is a disjunction  $\varphi_1 \vee \dots \vee \varphi_n$  of FO-formulas, with at most one of them being an atom and the remaining ones negations of atoms [52]. A *Horn formula* is constructed from basic Horn formulas using  $\wedge$ ,  $\exists$ , and  $\forall$ .

**Theorem 1.** (i) Every  $\text{horn}\mathcal{ALC}$ -concept is equivalent to a Horn formula with one free variable and every  $\text{horn}\mathcal{ALC}$ -CI is equivalent to a Horn sentence.

(ii) There exists an  $\mathcal{ALC}$ -concept (TBox) that is equivalent to a Horn formula (Horn sentence), but not equivalent to any  $\text{horn}\mathcal{ALC}$ -concept (TBox).

*Proof.* (i) is proved by a straightforward induction on the construction of  $\text{horn}\mathcal{ALC}$ -concepts. To prove (ii), consider first the  $\mathcal{ALC}$ -concept

$$C_{\nabla} = (\exists R. \top \sqcap \forall R.A) \rightarrow B.$$

It is not hard to check that  $C_{\nabla}^{\dagger}$  has the same models as

$$\exists y R(x, y) \rightarrow \exists z (R(x, z) \wedge (A(z) \rightarrow B(x))),$$

which is equivalent to a Horn formula. Example 1 below shows that  $C_{\nabla}$  is not equivalent to any  $\text{horn}\mathcal{ALC}$ -concept.

Next, consider the  $\mathcal{ALC}$ -TBox  $\mathcal{T}_{\text{horn}}$  with the following CIs:

$$\begin{aligned} E &\sqsubseteq A_1 \sqcup A_2 \sqcup \exists R.(\neg B_1 \sqcap \neg B_2), & \exists R.(B_1 \sqcap B_2) &\sqsubseteq \perp, \\ E &\sqsubseteq \exists R.\top, & \exists R.B_1 &\sqsubseteq \exists R.B_2, & \exists R.B_2 &\sqsubseteq \exists R.B_1. \end{aligned}$$

The FO-translations of all of them but the first one are obviously (equivalent to) Horn sentences. We take a conjunction of these translations together with the sentence

$$\forall x [E(x) \rightarrow \exists y (R(x, y) \wedge (B_1(y) \rightarrow A_1(x)) \wedge (B_2(y) \rightarrow A_2(x)))],$$

which is also equivalent to a Horn one. One can check that the resulting sentence is equivalent to  $\mathcal{T}_{\text{horn}}$ . On the other hand, Example 1 below shows that  $\mathcal{T}_{\text{horn}}$  is not equivalent to any  $\text{horn}\mathcal{ALC}$ -TBox.  $\square$

Given Theorem 1, a natural question arises whether it is possible to design a syntactic extension of  $\text{horn}\mathcal{ALC}$  that captures the intersection of  $\mathcal{ALC}$  and Horn FO. We discuss this problem in the conclusion of this paper.

We remind the reader of two usual operations on structures. The *product*  $\prod_{i \in I} \mathfrak{A}_i$  of a family of  $\tau$ -structures  $\mathfrak{A}_i$ ,  $i \in I$ ,

is defined as follows: its domain  $\text{dom}(\prod_{i \in I} \mathfrak{A}_i)$  is the set of functions  $f: I \rightarrow \bigcup_{i \in I} \text{dom}(\mathfrak{A}_i)$  with  $f(i) \in \text{dom}(\mathfrak{A}_i)$ , for  $i \in I$ , and

$$A^{\prod_{i \in I} \mathfrak{A}_i} = \{f \in \text{dom}(\prod_{i \in I} \mathfrak{A}_i) \mid \forall i \in I f(i) \in A^{\mathfrak{A}_i}\},$$

$$R^{\prod_{i \in I} \mathfrak{A}_i} = \{(f, g) \in (\text{dom}(\prod_{i \in I} \mathfrak{A}_i))^2 \mid \forall i \in I (f(i), g(i)) \in R^{\mathfrak{A}_i}\}.$$

Horn formulas are *preserved under products* in the sense that

$$\forall i \in I \mathfrak{A}_i \models \varphi(f_1(i), \dots, f_n(i)) \Rightarrow \prod_{i \in I} \mathfrak{A}_i \models \varphi(f_1, \dots, f_n)$$

for all Horn formulas  $\varphi$ . Note that an FO-formula is equivalent to a Horn formula iff it is preserved under the more general *reduced products* (modulo filters over  $I$ ) [52].

The *disjoint union*  $\mathfrak{A}$  of a family  $\mathfrak{A}_i$ ,  $i \in I$ , of structures has domain  $\bigcup_{i \in I} \text{dom}(\mathfrak{A}_i) \times \{i\}$  and

$$\begin{aligned} A^{\mathfrak{A}} &= \{(a, i) \mid a \in A^{\mathfrak{A}_i}\}, \\ R^{\mathfrak{A}} &= \{((a, i), (b, i)) \mid (a, b) \in R^{\mathfrak{A}_i}\}. \end{aligned}$$

$\mathcal{ALC}$ -TBoxes  $\mathcal{T}$  are *invariant under disjoint unions*, that is:

$$\forall i \in I \mathfrak{A}_i \models \mathcal{T} \Leftrightarrow \mathfrak{A} \models \mathcal{T}.$$

By the *depth* of a concept  $C$  we mean the maximal number of nestings of  $\exists R$  and  $\forall R$  in  $C$ . For example, the concepts  $\exists R.\exists R.A$  and  $\exists R.\forall R.A$  are of depth 2. The *depth* of a TBox is the maximum over the depths of the concepts occurring in it. By a *pointed structure* we mean a pair  $\mathfrak{A}, X$  with a structure  $\mathfrak{A}$  and a non-empty set  $X \subseteq \text{dom}(\mathfrak{A})$ . If  $X = \{a\}$ , we simply write  $\mathfrak{A}, a$ .

**Definition 2 (DL indistinguishability).** For any DL  $\mathcal{L}$ ,  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$ ,  $a \in \text{dom}(\mathfrak{A})$ ,  $X \subseteq \text{dom}(\mathfrak{A})$ ,  $b \in \text{dom}(\mathfrak{B})$ , and  $\ell < \omega$ , we write:

- $\mathfrak{A}, X \leq_{\mathcal{L}}^{(\ell)} \mathfrak{B}, b$  if  $X \subseteq C^{\mathfrak{A}}$  implies  $b \in C^{\mathfrak{B}}$ , for any  $\mathcal{L}$ -concept  $C$  (of depth  $\leq \ell$ );
- $\mathfrak{A}, a \equiv_{\mathcal{L}}^{(\ell)} \mathfrak{B}, b$  if  $\mathfrak{A}, a \leq_{\mathcal{L}}^{(\ell)} \mathfrak{B}, b$  and  $\mathfrak{B}, b \leq_{\mathcal{L}}^{(\ell)} \mathfrak{A}, a$ ;
- $\mathfrak{A} \leq_{\mathcal{L}} \mathfrak{B}$  if  $\mathfrak{A} \models C \sqsubseteq D$  implies  $\mathfrak{B} \models C \sqsubseteq D$ , for any  $\mathcal{L}$ -CI  $C \sqsubseteq D$ ;
- $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$  if  $\mathfrak{A} \leq_{\mathcal{L}} \mathfrak{B}$  and  $\mathfrak{B} \leq_{\mathcal{L}} \mathfrak{A}$ .

We now recall the model comparison games for indistinguishability in  $\mathcal{ELU}$  that will be required as part of the model comparison games for  $\text{horn}\mathcal{ALC}$ . The  $\mathcal{ELU}$  case is rather straightforward and folklore [59], but it will remind the reader of the basics of model comparison games used in this paper.

**Definition 3 (simulation).** A relation  $S \subseteq \text{dom}(\mathfrak{A}) \times \text{dom}(\mathfrak{B})$  is a *simulation* between  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$  if the following conditions hold:

- (*atom*) for any  $A \in \tau$ , if  $(a, b) \in S$  and  $a \in A^{\mathfrak{A}}$ , then  $b \in A^{\mathfrak{B}}$ ,
- (*forth*) for any  $R \in \tau$ , if  $(a, b) \in S$  and  $(a, a') \in R^{\mathfrak{A}}$ , then there is  $b'$  with  $(b, b') \in R^{\mathfrak{B}}$  and  $(a', b') \in S$ .

We write  $\mathfrak{A}, a \preceq_{\text{sim}} \mathfrak{B}, b$  if there exists a simulation  $S$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  with  $(a, b) \in S$ .

Simulations can be equivalently described as games between two players on the disjoint union of  $\mathfrak{A}$  and  $\mathfrak{B}$ . A position in the *simulation game* is a pair of nodes  $(a, b) \in \text{dom}(\mathfrak{A}) \times \text{dom}(\mathfrak{B})$ , marked by pebbles. The players move, in turns, the pebbles along binary relations in  $\tau$ . The first player chooses an  $R \in \tau$  and moves the pebble in  $\mathfrak{A}$  along  $R^{\mathfrak{A}}$ , the second player must respond in  $\mathfrak{B}$  complying with  $(atom_f)$  and  $(forth)$ . The second player wins a game starting at  $(a, b)$  if she can always respond to the first player's moves, *ad infinitum*. One can show that the second player has a winning strategy iff  $\mathfrak{A}, a \preceq_{sim} \mathfrak{B}, b$ .

Besides the infinitary simulation games corresponding to Definition 3, we consider games with a fixed number  $\ell$  of moves. We write  $\mathfrak{A}, a \preceq_{sim}^{\ell} \mathfrak{B}, b$  if the second player has a winning strategy in the simulation game with  $\ell$  rounds starting from  $(a, b)$ . We write  $\mathfrak{A}, a \preceq_{sim}^{\omega} \mathfrak{B}, b$  if  $\mathfrak{A}, a \preceq_{sim}^{\ell} \mathfrak{B}, b$  for every  $\ell < \omega$ .

**Theorem 2 (Ehrenfeucht-Fraïssé game for  $\mathcal{ELU}$ ).** *For any finite vocabulary  $\tau$ , pointed  $\tau$ -structures  $\mathfrak{A}, a$  and  $\mathfrak{B}, b$ , and any  $\ell < \omega$ , we have*

$$\mathfrak{A}, a \preceq_{\mathcal{ELU}}^{\ell} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}, a \preceq_{sim}^{\ell} \mathfrak{B}, b.$$

Thus,  $\mathfrak{A}, a \preceq_{\mathcal{ELU}} \mathfrak{B}, b$  iff  $\mathfrak{A}, a \preceq_{sim}^{\omega} \mathfrak{B}, b$ . If  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite, then

$$\mathfrak{A}, a \preceq_{\mathcal{ELU}} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}, a \preceq_{sim} \mathfrak{B}, b.$$

In some proofs, we shall also be using bisimulations. Recall that a relation  $S$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  is a *bisimulation* if  $S$  is a simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ , and its inverse is a simulation between  $\mathfrak{B}$  and  $\mathfrak{A}$ . The notion of  $\ell$ -bisimilarity is defined by restricting the corresponding *bisimulation game* to  $\ell$  moves. This notion characterizes indistinguishability in  $\mathcal{ALC}$ : pointed structures  $\mathfrak{A}, a$  and  $\mathfrak{B}, b$  are  $\ell$ -bisimilar iff  $\mathfrak{A}, a \equiv_{\mathcal{ALC}}^{\ell} \mathfrak{B}, b$  [33].

### III. SIMULATIONS FOR *hornALC*

We now define a new model comparison game, the *Horn simulation game*, and prove that it provides an Ehrenfeucht-Fraïssé characterization of the relation  $\mathfrak{A}, a \preceq_{hornALC} \mathfrak{B}, b$ . As *hornALC* is not closed under negation, Horn simulations will be non-symmetric. Moreover, since *hornALC* is not closed under disjunction, Horn simulations relate non-empty *sets* of elements from  $\text{dom}(\mathfrak{A})$  with elements from  $\text{dom}(\mathfrak{B})$ . Thus, rather than characterizing  $\mathfrak{A}, a \preceq_{hornALC} \mathfrak{B}, b$  only, we actually characterize the relation  $\mathfrak{A}, X \preceq_{hornALC} \mathfrak{B}, b$ . To define the relation between subsets of  $\text{dom}(\mathfrak{A})$  along which the pebble is moved in the Horn simulation game, we set  $XR^{\uparrow}Y$ , for a binary relation  $R$  and sets  $X, Y$ , if for any  $a \in X$ , there exists  $b \in Y$  with  $(a, b) \in R$ , and we set  $XR^{\downarrow}Y$  if, for any  $b \in Y$ , there exists  $a \in X$  with  $(a, b) \in R$ .

**Definition 4 (Horn simulation).** A *Horn simulation* between  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$  is a relation  $Z \subseteq \mathcal{P}(\text{dom}(\mathfrak{A})) \times \text{dom}(\mathfrak{B})$  such that  $(X, b) \in Z$  implies  $X \neq \emptyset$  and the following hold:

$(atom_h)$  for any  $A \in \tau$ , if  $(X, b) \in Z$  and  $X \subseteq A^{\mathfrak{A}}$ , then  $b \in A^{\mathfrak{B}}$ ;

$(forth_h)$  for any  $R \in \tau$ , if  $(X, b) \in Z$  and  $XR^{\uparrow}Y$ , then there exist  $Y' \subseteq Y$  and  $b' \in \text{dom}(\mathfrak{B})$  with  $(b, b') \in R^{\mathfrak{B}}$  and  $(Y', b') \in Z$ ;

$(back_h)$  for any  $R \in \tau$ , if  $(X, b) \in Z$  and  $(b, b') \in R^{\mathfrak{B}}$ , then there is  $Y \subseteq \text{dom}(\mathfrak{A})$  with  $XR^{\downarrow}Y$  and  $(Y, b') \in Z$ ;

$(sim)$  if  $(X, b) \in Z$ , then  $\mathfrak{B}, b \preceq_{sim} \mathfrak{A}, a$  for every  $a \in X$ .

We write  $\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b$  if there exists a Horn simulation  $Z$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  such that  $(X, b) \in Z$ .

Condition  $(atom_h)$  ensures that concept names are preserved under Horn simulations, and conditions  $(forth_h)$ ,  $(back_h)$ , and  $(sim)$  ensure, recursively, the preservation of concepts of the form  $\exists R.H$ ,  $\forall R.H$  and  $L \rightarrow H$ , respectively. Note that  $(sim)$  implies that the converse of  $(atom_h)$  holds as well, and so  $\mathfrak{A}, X \preceq_{horn} \mathfrak{B}, b$  entails  $X \subseteq A^{\mathfrak{A}}$  iff  $b \in A^{\mathfrak{B}}$ , for all  $A \in \tau$ , which reflects that  $A \rightarrow \perp$  is a *hornALC*-concept.

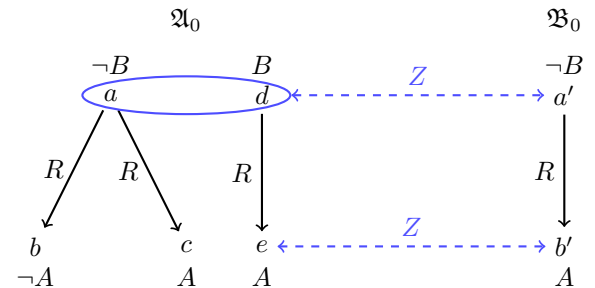
As we intend Horn simulations to characterize *hornALC*-concepts, which are  $\mathcal{ALC}$ -concepts, Horn simulations should subsume bisimulations. The following lemma states that this is indeed the case. It also shows that having *sets* as the first component of positions in the Horn simulation games is the defining difference between Horn simulations and bisimulations. The (straightforward) proof is instructive to understand Horn simulations.

**Lemma 1.** (i) *If  $Z$  is a bisimulation between  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$ , then  $\{(\{a\}, b) \mid (a, b) \in Z\}$  is a Horn simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ .* (ii) *Conversely, if  $Z$  is a Horn simulation with a singleton  $X$  in every  $(X, b) \in Z$ , then  $\{(a, b) \mid (\{a\}, b) \in Z\}$  is a bisimulation between  $\mathfrak{A}$  and  $\mathfrak{B}$ .*

As the FO-translations of *hornALC*-concepts are Horn formulas, and the Horn formulas are (almost) characterized as the fragment of FO preserved under products, one could expect products to be closely related to Horn simulations. We now show this to be the case. Consider a family of  $\tau$ -structures  $\mathfrak{A}_i$ ,  $i \in I$ , and let  $\mathfrak{A}$  be the disjoint union of the  $\mathfrak{A}_i$ . Define a relation  $Z$  between  $\mathcal{P}(\text{dom}(\mathfrak{A}))$  and  $\prod_{i \in I} \mathfrak{A}_i$  by setting  $(Y, f) \in Z$  if  $Y \subseteq \text{dom}(\mathfrak{A})$ ,  $f \in \text{dom}(\prod_{i \in I} \mathfrak{A}_i)$ , and  $\text{dom}(\mathfrak{A}_i) \cap Y = \{f(i)\}$  for all  $i \in I$ . The proof of the following is again straightforward and instructive.

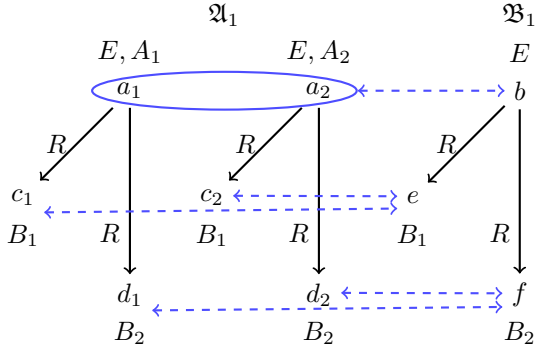
**Lemma 2.**  *$Z$  is a Horn simulation between  $\mathfrak{A}$  and  $\prod_{i \in I} \mathfrak{A}_i$ .*

The following examples illustrate that Horn simulations can be seen as a proper generalization of both bisimulations and products.



**Example 1.** (i) Let  $\mathfrak{A}_0$  and  $\mathfrak{B}_0$  be the structures shown above.  $Z = \{(\{a, d\}, a'), (\{e\}, b')\}$  is a Horn simulation between  $\mathfrak{A}_0$  and  $\mathfrak{B}_0$ . For the concept  $C_\nabla$  from Theorem 1, we have  $\{a, d\} \subseteq C_\nabla^{\mathfrak{A}_0}$  but  $a' \notin C_\nabla^{\mathfrak{B}_0}$ . Thus, by Lemma 3 below,  $C_\nabla$  is not equivalent to any  $\text{hornALCC}$ -concept.

(ii) Let  $\mathfrak{A}_1$  and  $\mathfrak{B}_1$  be the structures below. Then the relation  $Z = \{(\{a_1, a_2\}, b), (\{c_1\}, e), (\{c_2\}, e), (\{d_1\}, f), (\{d_2\}, f)\}$  is a surjective Horn simulation between  $\mathfrak{A}_1$  and  $\mathfrak{B}_1$ . Moreover, we have  $\mathfrak{A}_1 \models \mathcal{T}_{\text{horn}}$  but  $\mathfrak{B}_1 \not\models \mathcal{T}_{\text{horn}}$  for the TBox  $\mathcal{T}_{\text{horn}}$  from Theorem 1. By Theorem 4 below,  $\mathcal{T}_{\text{horn}}$  is not equivalent to any  $\text{hornALCC}$ -TBox.



Given the notion of Horn simulation, we next introduce the *Horn simulation game* between  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$  in the expected way. In the infinite case, it consists of the following nested games. Using simulation games, one can check whether condition (*sim*) holds for a pair  $(X, b)$ . Then, in the main Horn simulation game, the second player must respond with pairs  $(Y', b')$  for (*forth*) and sets  $Y$  for (*back*) such that the new position satisfies (*sim*) and the remaining conditions of Definition 4. In the Horn simulation game with  $\ell$  rounds, some care must be taken: as we want to characterize the depth  $\ell$  fragment of  $\text{hornALCC}$ , we have to decompose condition (*sim*). Thus, define pairs  $(X, b)$  satisfying (*sim* <sup>$\ell$</sup> ) as those for which player 2 has a winning strategy for the  $\ell$ -round simulation game for all  $(b, a)$  with  $a \in X$ . Then, inductively, player 2 has a winning strategy in the  $(\ell + 1)$ -round Horn simulation game at position  $(X, b)$  if  $(X, b)$  satisfies (*sim* <sup>$\ell + 1$</sup> ) and player 2 can react to player 1's first move by choosing  $Y$  in such a way that condition (*sim* <sup>$\ell$</sup> ) holds for the resulting position and she has a winning strategy in the resulting  $\ell$ -round game. A formal definition is given in the full version. We write  $\mathfrak{A}, X \preceq_{\text{horn}}^{\ell} \mathfrak{B}, b$  if player 2 has a winning strategy in the  $\ell$ -round game.

**Theorem 3 (Ehrenfeucht-Fraïssé game for  $\text{hornALCC}$ ).** For any finite vocabulary  $\tau$ , pointed  $\tau$ -structures  $\mathfrak{A}, a$  and  $\mathfrak{B}, b$ , and any  $\ell < \omega$ , we have

$$\mathfrak{A}, a \leq_{\text{hornALCC}}^{\ell} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}, a \preceq_{\text{horn}}^{\ell} \mathfrak{B}, b.$$

Thus,  $\mathfrak{A}, a \leq_{\text{hornALCC}} \mathfrak{B}, b$  iff  $\mathfrak{A}, a \preceq_{\text{horn}}^{\omega} \mathfrak{B}, b$ . If  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite, then

$$\mathfrak{A}, a \leq_{\text{hornALCC}} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}, a \preceq_{\text{horn}} \mathfrak{B}, b.$$

As discussed above, to prove Theorem 3, we actually show the following stronger statement:

**Lemma 3.** For any finite vocabulary  $\tau$ , pointed  $\tau$ -structures  $\mathfrak{A}, X$  and  $\mathfrak{B}, b$ , and any  $\ell < \omega$ ,

$$\mathfrak{A}, X \leq_{\text{hornALCC}}^{\ell} \mathfrak{B}, b \quad \text{iff} \quad \exists X_0 \subseteq X \quad \mathfrak{A}, X_0 \preceq_{\text{horn}}^{\ell} \mathfrak{B}, b.$$

If  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite, then

$$\mathfrak{A}, X \leq_{\text{hornALCC}} \mathfrak{B}, b \quad \text{iff} \quad \exists X_0 \subseteq X \quad \mathfrak{A}, X_0 \preceq_{\text{horn}} \mathfrak{B}, b.$$

*Proof.* (sketch) The second claim follows directly from the first one, which we prove here. For any  $\ell < \omega$  and any pointed  $\tau$ -structure  $\mathfrak{A}, a$ , let  $\lambda_{\mathfrak{A}, \ell, a}$  be an  $\mathcal{ELU}[\tau]$ -concept of depth  $\leq \ell$  such that, for any pointed  $\tau$ -structure  $\mathfrak{B}, b$ ,

$$b \in \lambda_{\mathfrak{A}, \ell, a}^{\mathfrak{B}} \quad \text{iff} \quad \mathfrak{A}, a \preceq_{\text{sim}}^{\ell} \mathfrak{B}, b. \quad (1)$$

The existence of  $\lambda_{\mathfrak{A}, \ell, a}$  follows immediately from the fact that there are only finitely-many non-equivalent  $\mathcal{ELU}[\tau]$ -concepts of any fixed depth  $\ell$ . Similarly, fix a finite set  $\text{Horn}_{\ell}$  of  $\text{hornALCC}$ -concepts of depth  $\leq \ell$  such that every  $\text{hornALCC}$ -concept of depth  $\leq \ell$  is equivalent to some concept in  $\text{Horn}_{\ell}$ . For a pointed  $\tau$ -structure  $\mathfrak{A}, X$ , let  $\rho_{\mathfrak{A}, \ell, X}$  be the conjunction of all concepts  $C$  in  $\text{Horn}_{\ell}$  with  $X \subseteq C^{\mathfrak{A}}$ . Clearly, we have

$$b \in \rho_{\mathfrak{A}, \ell, X}^{\mathfrak{B}} \quad \text{iff} \quad \mathfrak{A}, X \leq_{\text{hornALCC}}^{\ell} \mathfrak{B}, b. \quad (2)$$

To prove the implication ( $\Rightarrow$ ) of the first claim, we define relations  $Z_{\ell} \subseteq \mathcal{P}(\text{dom}(\mathfrak{A})) \times \text{dom}(\mathfrak{B})$ ,  $\ell < \omega$ , by setting  $(X, b) \in Z_{\ell}$  if  $X \neq \emptyset$  and the following two conditions hold:

- (i)  $b \in \rho_{\mathfrak{A}, \ell, X}^{\mathfrak{B}}$ ,
- (ii)  $X \subseteq \lambda_{\mathfrak{B}, \ell, b}^{\mathfrak{A}}$ .

**Claim 1.** For any  $\ell < \omega$ ,  $\emptyset \neq X \subseteq \text{dom}(\mathfrak{A})$  and  $b \in \text{dom}(\mathfrak{B})$ , if  $(X, b) \in Z_{\ell}$ , then  $\mathfrak{A}, X \preceq_{\text{horn}}^{\ell} \mathfrak{B}, b$ .

*Proof of claim.* We proceed by induction on  $\ell < \omega$ . The basis  $\ell = 0$  holds by definition. So suppose that Claim 1 has been proved for  $\ell$  and that  $(X, b) \in Z_{\ell+1}$ . We show  $\mathfrak{A}, X \preceq_{\text{horn}}^{\ell+1} \mathfrak{B}, b$ . Condition (*atom*) holds by definition. For (*forth*), suppose player 1 moves the pebble to  $Y$  with  $X R^{\mathfrak{A}} \uparrow Y$ . Then  $X \subseteq (\exists R. \rho_{\mathfrak{A}, \ell, Y})^{\mathfrak{A}}$ . By the definition of  $Z_{\ell+1}$ ,  $b \in (\exists R. \rho_{\mathfrak{A}, \ell, Y})^{\mathfrak{B}}$ . Let player 2 respond with  $b'$  and  $Y'$  such that  $(b, b') \in R^{\mathfrak{B}}$ ,  $b' \in \rho_{\mathfrak{A}, \ell, Y}^{\mathfrak{B}}$ , and

$$Y' = Y \cap \lambda_{\mathfrak{B}, \ell, b'}^{\mathfrak{A}}.$$

We show that  $(Y', b')$  is as required for (*forth*). By IH, it suffices to prove that  $(Y', b') \in Z_{\ell}$ . To show  $Y' \neq \emptyset$ , suppose otherwise. Then  $Y \subseteq (\lambda_{\mathfrak{B}, \ell, b'} \rightarrow \perp)^{\mathfrak{A}}$ , so  $(\lambda_{\mathfrak{B}, \ell, b'} \rightarrow \perp)$  is equivalent to a conjunct of  $\rho_{\mathfrak{A}, \ell, Y}$ . By the construction of  $b'$ , we have  $b' \in (\lambda_{\mathfrak{B}, \ell, b'} \rightarrow \perp)^{\mathfrak{B}}$ . On the other hand,  $b' \in \lambda_{\mathfrak{B}, \ell, b'}^{\mathfrak{B}}$ , which is impossible.

Condition (i) is proved similarly and condition (ii) holds by the definition of  $Y'$ .

For (*back*), suppose player 1 moves the pebble to  $b'$  with  $(b, b') \in R^{\mathfrak{B}}$ . For every  $C \in \text{Horn}_{\ell}$  with  $b' \notin C^{\mathfrak{B}}$ , take some  $a_C \in X$  and  $a'_C$  with  $(a_C, a'_C) \in R^{\mathfrak{A}}$  such that  $a'_C \in (\lambda_{\mathfrak{B}, \ell, b'} \cap \neg C)^{\mathfrak{A}}$ . They exist since otherwise we would have  $X \subseteq (\forall R. (\lambda_{\mathfrak{B}, \ell, b'} \rightarrow C))^{\mathfrak{A}}$  but  $b \notin (\forall R. (\lambda_{\mathfrak{B}, \ell, b'} \rightarrow C))^{\mathfrak{B}}$ , which contradicts the definition of  $Z_{\ell+1}$ . Now let player 2 respond with the set  $Y$  of all such  $a'_C$ . Then  $X R^{\mathfrak{A}} \uparrow Y$  and



$(Y, b') \in Z_\ell$  as  $b' \in \rho_{\mathfrak{A}, \ell, Y}^{\mathfrak{B}}$  and  $Y \subseteq \lambda_{\mathfrak{B}, \ell, b'}^{\mathfrak{B}}$  hold by the construction of  $Y$ . By IH,  $\mathfrak{A}, Y \preceq_{\text{horn}}^\ell \mathfrak{B}, b'$ , as required.

Finally,  $(\text{sim}^{\ell+1})$  follows from (1), which completes the proof of Claim 1.

Now assume that  $\mathfrak{A}, X \leq_{\text{horn}, \mathcal{ALC}}^\ell \mathfrak{B}, b$ . Then it suffices to prove that if  $b \in \rho_{\mathfrak{A}, \ell, X}^{\mathfrak{B}}$ , then there exists  $X_0 \subseteq X$  with  $(X_0, b) \in Z_\ell$ . But for  $X_0 = X \cap \lambda_{\mathfrak{B}, \ell, b}^{\mathfrak{A}}$  this can be proved in the same way as Claim 1 above.

The proof of the implication  $(\Leftarrow)$  of the first claim is by induction on  $\ell < \omega$ .  $\square$

Horn simulations can also characterize *hornALC*-TBoxes. For  $\ell < \omega$ , we write  $\mathfrak{A} \preceq_{\text{horn}}^{(\ell)} \mathfrak{B}$  if, for every  $b \in \text{dom}(\mathfrak{B})$ , there exists  $X \subseteq \text{dom}(\mathfrak{A})$  such that  $\mathfrak{A}, X \preceq_{\text{horn}}^{(\ell)} \mathfrak{B}, b$ .

**Theorem 4.** *For any finite vocabulary  $\tau$ ,  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$ , and  $\ell < \omega$ , we have*

$$\mathfrak{A} \leq_{\text{horn}, \mathcal{ALC}}^\ell \mathfrak{B} \quad \text{iff} \quad \mathfrak{A} \preceq_{\text{horn}}^\ell \mathfrak{B}.$$

If  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite, then

$$\mathfrak{A} \leq_{\text{horn}, \mathcal{ALC}} \mathfrak{B} \quad \text{iff} \quad \mathfrak{A} \preceq_{\text{horn}} \mathfrak{B}.$$

#### IV. COMPLEXITY OF MODEL INDISTINGUISHABILITY

We next study the complexity of deciding the relations  $\leq_{\text{horn}, \mathcal{ALC}}$  and  $\equiv_{\text{horn}, \mathcal{ALC}}$  and their restrictions  $\leq_{\text{horn}, \mathcal{ALC}}^\ell$  and  $\equiv_{\text{horn}, \mathcal{ALC}}^\ell$  on the level of concepts. The related problems *on the TBox level* (cf. Theorem 4) have the same complexity as shown in the full version. We refer to the respective decision problems as  $(\ell)$ -*entailment* and  $(\ell)$ -*equivalence*; for instance, entailment is the problem of deciding whether  $\mathfrak{A}, a \leq_{\text{horn}, \mathcal{ALC}} \mathfrak{B}, b$  for input  $\mathfrak{A}, \mathfrak{B}, a, b$ , and  $\ell$ -equivalence is the problem of deciding whether  $\mathfrak{A}, a \equiv_{\text{horn}, \mathcal{ALC}} \mathfrak{B}, b$  for input  $\mathfrak{A}, \mathfrak{B}, a, b, \ell$ .

As a second application of the Ehrenfeucht-Fraïssé results we investigate *concept learning by example (CBE)*. CBE is a supervised learning problem with applications in knowledge engineering for automatically deriving new and potentially interesting concept descriptions from labelled data. Intuitively, given some relational data and sets of positive and negative examples, the goal is to find a concept that generalizes the positive examples, but avoids the negative ones. The associated decision problem is formally defined as follows:

**Input:** structure  $\mathfrak{A}$ , positive and negative examples  $P, N$ .

**Question:** is there a *hornALC*-concept  $C$  with  $P \subseteq C^{\mathfrak{A}}$  and  $N \cap C^{\mathfrak{A}} = \emptyset$ ?

The connection to Horn simulations is given by Lemma 3: an input  $\mathfrak{A}, P, N$  is a yes-instance of CBE iff  $\mathfrak{A}, P \not\leq_{\text{horn}} \mathfrak{A}, b$  for all  $b \in N$ . We denote by  $\ell$ -CBE the variant of CBE restricted to *hornALC*-concepts of depth  $\ell$ . This variant is important in practice as the user is interested in small separating concepts.

Our main result in this section is the following theorem:

**Theorem 5.** *Entailment, equivalence, and CBE are EXPTIME-complete. Moreover,  $\ell$ -entailment,  $\ell$ -equivalence and  $\ell$ -CBE are EXPTIME-complete if  $\ell$  is given in binary and PSPACE-complete if  $\ell$  is given in unary.*

It is worth mentioning the striking contrast between this EXPTIME result and the fact that the same problems for  $\mathcal{ALC}$  are in PTIME. On the one hand, the EXPTIME lower bounds provide evidence that the use of sets in the notion of Horn simulations is inevitable. On the other hand, observe that CBE for  $\mathcal{ALC}$  is in PTIME because there is an  $\mathcal{ALC}$ -concept separating the positive and negative examples iff  $\mathfrak{A}, a$  and  $\mathfrak{A}, b$  are not bisimilar, for any  $a \in P$  and  $b \in N$ . Consequently, the positive examples can be treated essentially separately and a naive application of this leads to overfitting, that is, the intended generalization of the positive examples is not taking place [26]. Thus, we can regard the EXPTIME result for CBE in *hornALC* as the price for obtaining real generalizations.

To prove Theorem 5, we focus on the unrestricted case. The following lemma gives complexity-theoretic reductions between the mentioned problems and the problem *HornSim* of deciding whether  $\mathfrak{A}, X \preceq_{\text{horn}} \mathfrak{B}, b$  for input  $\mathfrak{A}, \mathfrak{B}, X, b$ .

**Lemma 4.** (1)  $\text{CBE} \leq_T^P \text{HornSim}$ ;

(2)  $\overline{\text{HornSim}} \leq_m^P \text{CBE}$ ;

(3)  $\text{HornSim} \leq_m^P \text{Entailment}$ ;

(4)  $\text{Entailment} \leq_m^P \text{HornSim}$ ;

(5)  $\text{Equivalence} \leq_T^P \text{Entailment}$ ;

(6)  $\text{Entailment} \leq_m^P \text{Equivalence}$ .

*Proof.* Here, we only show the most interesting reduction (3).

Let  $\mathfrak{A}, \mathfrak{B}, X, b$  be the input to *HornSim*. Define  $\mathfrak{A}'$  by adding a new  $R$ -predecessor  $a$  to all nodes in  $X$ . Further, define  $\mathfrak{B}'$  by taking the disjoint union of  $\mathfrak{A}$  and  $\mathfrak{B}$  and adding a new  $R$ -predecessor  $d$  to  $b$ , and making  $d$  also a predecessor of all nodes in (the copy of)  $X$ . Then we have

$$\mathfrak{A}, X \preceq_{\text{horn}} \mathfrak{B}, b \quad \text{iff} \quad \mathfrak{A}', a \preceq_{\text{horn}} \mathfrak{B}', d,$$

which is equivalent to  $\mathfrak{A}', a \leq_{\text{horn}, \mathcal{ALC}} \mathfrak{B}', d$  by Theorem 3.  $\square$

Thus, to prove Theorem 5, it suffices to show the following:

**Lemma 5.** *HornSim is EXPTIME-complete.*

For the upper bound, we observe that the Horn simulation game can be implemented by an alternating Turing machine (ATM) using only polynomial space. For the lower bound, we carefully adapt a strategy from [60] for proving that the simulation problem between two structures  $\mathfrak{A}$  and  $\mathfrak{B}$  is EXPTIME-hard when  $\mathfrak{A}$  is given as a *fair concurrent transition system*, that is, a certain synchronized product of structures. More precisely, we reduce the word problem of polynomially space-bounded ATMs. Let  $M$  be an  $s(n)$ -space bounded ATM and  $w$  an input of length  $n$ . We construct structures  $\mathfrak{A}, \mathfrak{B}$ ,  $X \subseteq \text{dom}(\mathfrak{A})$ , and  $b \in \text{dom}(\mathfrak{B})$  such that

$$M \text{ accepts } w \quad \text{iff} \quad \mathfrak{A}, X \preceq_{\text{horn}} \mathfrak{B}, b.$$

The structure  $\mathfrak{A}$  can be thought of as the disjoint union of  $s(n)$  structures  $\mathfrak{A}_1, \dots, \mathfrak{A}_{s(n)}$  and a single copy of  $\mathfrak{B}$  (plus some connections from the  $\mathfrak{A}_i$  to  $\mathfrak{B}$ ). Intuitively, each sub-structure  $\mathfrak{A}_i$  is responsible for tape cell  $i$  of one of  $M$ 's configurations on input  $w$ ; thus, the domain of each  $\mathfrak{A}_i$  consists of the possible contents of a single cell. As usual, the challenge is



the synchronization. Here, different tape cells are synchronized via the simulation conditions using different role names: one role name  $R_{q,a,i,d}$  for every possible state  $q$  of  $M$ , current head position  $i$ , read symbol  $a$ , and branching direction  $d$  (we assume that  $M$  has binary branching). The extension of such a role name  $R_{q,a,i,d}$  in a structure  $\mathfrak{A}_j$  is defined in the obvious way, respecting  $M$ 's transition relation. The set  $X$  consists of  $M$ 's initial configuration on input  $w$ .

The role of  $\mathfrak{B}$  (as the second structure) is to control  $M$ 's computation; its domain is independent of  $M$  and consists of 20 elements only. It manages both the switch between universal and existential states and the intended acceptance value of the current configuration (1 if the configuration leads to acceptance, 0 otherwise). Universal and existential elements are fully interconnected with the mentioned role names  $R_{q,a,i,d}$ . The initial element  $b$  for the reduction is a universal element (without loss of generality  $M$ 's initial state is universal), and corresponds to acceptance value 1.

Now, conditions  $(atom_h)$  and  $(forth_h)$  are responsible for simulating  $M$ 's computation on input  $w$  and  $(atom_h)$  ensures that every reached set  $X$  in  $\mathfrak{A}$  corresponds to a valid configuration. Conditions  $(sim)$  and  $(back_h)$  do not have a real purpose for the reduction, but need to be reflected in the mentioned inclusion of (a copy of)  $\mathfrak{B}$  in  $\mathfrak{A}$ .

## V. EXPRESSIVE COMPLETENESS FOR $horn\mathcal{ALC}$

In this section, we prove that an FO-formula with one free variable is equivalent to a  $horn\mathcal{ALC}$ -concept just in case it is preserved under Horn simulations, and that an FO-sentence is equivalent to a  $horn\mathcal{ALC}$ -TBox just in case it is invariant under disjoint unions and preserved under *global* (that is, surjective) Horn simulations. We prove these results both in the classical setting defined above and in the finite model theory setting where the notions of equivalence, preservation, and invariance are relativized to finite models.

In the concept case, by the van Benthem-Rosen characterization of  $\mathcal{ALC}$ -concepts as the bisimulation invariant fragment of FO and since preservation under Horn simulations implies invariance under bisimulations (by Lemma 1), it suffices to prove that an  $\mathcal{ALC}$ -concept is equivalent to a  $horn\mathcal{ALC}$ -concept iff it is preserved under Horn simulations (in the classical and finite model theory setting). We will, therefore, formulate the expressive completeness result within  $\mathcal{ALC}$ . In the TBox case, it is known that an FO-sentence is equivalent to an  $\mathcal{ALC}$ -TBox just in case it is invariant under disjoint unions and preserved under bisimulations [35], [45]; thus, again by Lemma 1, it suffices to show that an  $\mathcal{ALC}$ -TBox is equivalent to a  $horn\mathcal{ALC}$ -TBox iff it is preserved under global Horn simulations. In both proofs, we employ Otto's finitary method [35] and show that, for  $\mathcal{ALC}$ -concepts (TBoxes) of depth  $\leq \ell$ , preservation under (global) Horn simulations is equivalent to preservation under (global)  $\ell$ -Horn simulations, which is the same as equivalence to a  $horn\mathcal{ALC}$ -concept (TBox) of depth  $\leq \ell$ .

We start with the concept case. Say that an  $\mathcal{ALC}$ -concept  $C$  is *preserved under  $(\ell)$ -Horn simulations* if, for any pointed

structures  $\mathfrak{A}, X$  and  $\mathfrak{B}, b$ , whenever  $X \subseteq C^{\mathfrak{A}}$  and  $\mathfrak{A}, X \preceq_{horn}^{(\ell)} \mathfrak{B}, b$  then  $b \in C^{\mathfrak{B}}$ .

**Theorem 6 (expressive completeness:  $horn\mathcal{ALC}$ -concepts).** *Let  $C$  be an  $\mathcal{ALC}$ -concept of depth  $\ell$ . Then the following conditions are equivalent (in the classical and finite model theory setting):*

- (1)  $C$  is equivalent to a  $horn\mathcal{ALC}$ -concept,
- (2)  $C$  is preserved under Horn simulations,
- (3)  $C$  is preserved under  $\ell$ -Horn simulations,
- (4)  $C$  is equivalent to a  $horn\mathcal{ALC}$ -concept of depth  $\leq \ell$ .

*Proof.* (sketch) (1)  $\Rightarrow$  (2) follows from Lemma 3; (4)  $\Rightarrow$  (1) is trivial; (3)  $\Rightarrow$  (4) is straightforward and proved in the full version. We thus focus on (2)  $\Rightarrow$  (3).

A structure  $\mathfrak{A}$  is called *tree-shaped* if the directed graph  $G_{\mathfrak{A}} = (\text{dom}(\mathfrak{A}), E)$  with  $E = \bigcup_{R \in \tau} R^{\mathfrak{A}}$  is a directed tree and  $R^{\mathfrak{A}} \cap S^{\mathfrak{A}} = \emptyset$  for all distinct role names  $R$  and  $S$ . The root of  $G_{\mathfrak{A}}$  is called the *root* of  $\mathfrak{A}$ . The *depth* of  $a \in \text{dom}(\mathfrak{A})$  is the length of the path from the root of  $\mathfrak{A}$  to  $a$ ; the root of  $\mathfrak{A}$  has depth 0. The disjoint union of tree-shaped structures is a *forest*. Recall that every pointed  $\mathfrak{A}, a$  can be unravelled into a tree-shaped structure  $\mathfrak{A}^*$  with root  $a$  such that  $\mathfrak{A}, a$  and  $\mathfrak{A}^*, a$  are bisimilar [32]. Note that  $\mathfrak{A}^*$  is infinite (even for finite  $\mathfrak{A}$ ) if  $G_{\mathfrak{A}}$  contains a cycle. The finite model theory version of Theorem 6 is not affected as one only needs the unravelled tree-shaped structures up to a finite depth  $\ell$ .

Suppose  $C$  is an  $\mathcal{ALC}$ -concept of depth  $\leq \ell$  preserved under Horn simulations. Let  $\mathfrak{A}, X$  and  $\mathfrak{B}, b$  be pointed structures such that  $\mathfrak{A}, X \preceq_{horn}^{\ell} \mathfrak{B}, b$  and  $X \subseteq C^{\mathfrak{A}}$ . We have to show that  $b \in C^{\mathfrak{B}}$ . For every  $a \in X$ , take a tree-shaped pointed structure  $\mathfrak{A}_a, a$  bisimilar to  $\mathfrak{A}, a$ . Let  $\mathfrak{B}', b$  be a tree-shaped pointed structure bisimilar to  $\mathfrak{B}, b$ . Then  $\mathfrak{A}', X \preceq_{horn}^{\ell} \mathfrak{B}', b$  for the disjoint union  $\mathfrak{A}'$  of  $\mathfrak{A}_a, a \in X$ . By bisimulation invariance of  $\mathcal{ALC}$ -concepts, we have  $X \subseteq C^{\mathfrak{A}'}$  and it suffices to prove that  $b \in C^{\mathfrak{B}'}$ . Remove from  $\mathfrak{A}'$  and  $\mathfrak{B}'$  all nodes of depth  $> \ell$  and denote the resulting structures by  $\mathfrak{A}''$  and  $\mathfrak{B}''$ , respectively. As  $C$  is of depth  $\leq \ell$ , we have  $X \subseteq C^{\mathfrak{A}''}$  and it suffices to prove that  $b \in C^{\mathfrak{B}''}$ . Using  $\mathfrak{A}', X \preceq_{horn}^{\ell} \mathfrak{B}', b$ , it is straightforward to show that  $\mathfrak{A}'', X \preceq_{horn} \mathfrak{B}'', b$ . Then  $b \in C^{\mathfrak{B}''}$  follows from the preservation of  $C$  under Horn simulations.

For the finite model theory setting, observe that  $\mathfrak{A}''$  and  $\mathfrak{B}''$  are finite if  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite.  $\square$

We now consider the TBox case. We say that an  $\mathcal{ALC}$ -TBox is *preserved under global  $(\ell)$ -Horn simulations* if  $\mathfrak{A} \models \mathcal{T}$  and  $\mathfrak{A} \preceq_{horn}^{(\ell)} \mathfrak{B}$  imply  $\mathfrak{B} \models \mathcal{T}$ .

**Theorem 7 (expressive completeness:  $horn\mathcal{ALC}$ -TBoxes).** *For any  $\mathcal{ALC}$ -TBox  $\mathcal{T}$  of depth  $\ell$ , the following conditions are equivalent (in the classical and finite model theory setting):*

- (1)  $\mathcal{T}$  is equivalent to a  $horn\mathcal{ALC}$ -TBox;
- (2)  $\mathcal{T}$  is preserved under global Horn simulations;
- (3)  $\mathcal{T}$  is preserved under global  $\ell$ -Horn simulations;
- (4)  $\mathcal{T}$  is equivalent to a  $horn\mathcal{ALC}$ -TBox of depth  $\leq \ell$ .

*Proof.* (sketch) We use the notation from the previous proof and focus on (2)  $\Rightarrow$  (3), showing (3)  $\Rightarrow$  (4) in the full version.

We require *injective  $\ell$ -Horn simulations*, which are defined as follows. Let  $\mathfrak{A}$  be a forest and  $\mathfrak{B}$  a tree-shaped structure. A sequence  $H^0, \dots, H^\ell$  of relations between  $\mathcal{P}(\text{dom}(\mathfrak{A}))$  and  $\text{dom}(\mathfrak{B})$  is called an *injective  $\ell$ -Horn simulation* if for each  $(X, b) \in H^i$  all  $a \in X$  are of depth  $i$  in  $\mathfrak{A}$  and  $b$  is of depth  $i$  in  $\mathfrak{B}$ , and the following conditions hold:

- if  $(X, b) \in H^i$ , then  $\mathfrak{A}, X \preceq_{\text{horn}}^i \mathfrak{B}, b$ , for  $0 \leq i \leq \ell$ ;
- if  $(X, b) \in H^i$  and  $XR^{\mathfrak{A}^\dagger}Y$ , then there are  $Y' \subseteq Y$  and  $b' \in \text{dom}(\mathfrak{B})$  with  $(b, b') \in R^{\mathfrak{B}}$  and  $(Y', b') \in H^{i+1}$ , for all  $R \in \tau$  and  $0 \leq i < \ell$ ;
- if  $(X, b) \in H^i$  and  $(b, b') \in R^{\mathfrak{B}}$ , then there exists  $Y \subseteq \text{dom}(\mathfrak{A})$  such that  $XR^{\mathfrak{A}^\dagger}Y$  and  $(Y', b') \in H^{i+1}$ , for all  $R \in \tau$  and  $0 \leq i < \ell$ ;
- if  $(X_0, b), (X_1, b) \in H^i$ , then  $X_0 = X_1$ , for  $0 \leq i \leq \ell$ .

If  $\mathfrak{A}, X \preceq_{\text{horn}}^\ell \mathfrak{B}, b$ , we can take, for  $a \in X$ , a tree-shaped pointed structure  $\mathfrak{A}_a, a$  bisimilar to  $\mathfrak{A}, a$  and a tree-shaped pointed structure  $\mathfrak{B}', b$  bisimilar to  $\mathfrak{B}, b$ . Then  $\mathfrak{A}', X \preceq_{\text{horn}}^\ell \mathfrak{B}', b$  for the disjoint union  $\mathfrak{A}'$  of the  $\mathfrak{A}_a, a \in X$ . By duplicating successors in  $\mathfrak{B}'$  sufficiently often (possibly exponentially many times), we obtain a tree-shaped pointed structure  $\mathfrak{B}'', b$  bisimilar to  $\mathfrak{B}', b$  such that there is an injective  $\ell$ -Horn simulation  $H^0, \dots, H^\ell$  between  $\mathfrak{A}'$  and  $\mathfrak{B}''$  with  $(X, b) \in H^0$ .

Now suppose  $\mathcal{T}$  is preserved under global Horn simulations. Let  $\mathfrak{B}$  be a structure such that there exists a model  $\mathfrak{A}$  of  $\mathcal{T}$  with  $\mathfrak{A} \preceq_{\text{horn}}^\ell \mathfrak{B}$ . We have to show that  $\mathfrak{B}$  is a model of  $\mathcal{T}$ . Let  $b_0 \in \text{dom}(\mathfrak{B})$  be arbitrary. It suffices to show  $b_0 \in (\neg C \sqcup D)^{\mathfrak{B}}$  for all  $C \sqsubseteq D \in \mathcal{T}$ . Since  $\mathfrak{A} \preceq_{\text{horn}}^\ell \mathfrak{B}$ , there is a set  $X \subseteq \text{dom}(\mathfrak{A})$  with  $\mathfrak{A}, X \preceq_{\text{horn}}^\ell \mathfrak{B}, b_0$ . For every  $a \in X$ , take a tree-shaped pointed structure  $\mathfrak{A}_a, a$  bisimilar to  $\mathfrak{A}, a$ . By the observation above, we can take a tree-shaped pointed interpretation  $\mathfrak{B}', b_0$  bisimilar to  $\mathfrak{B}, b_0$  and the disjoint union  $\mathfrak{A}'$  of the  $\mathfrak{A}_a, a \in X$ , such that there is an injective  $\ell$ -Horn simulation  $H^0, \dots, H^\ell$  between  $\mathfrak{A}'$  and  $\mathfrak{B}'$  with  $(X, b_0) \in H^0$ . By bisimulation invariance of  $\mathcal{ALC}$ -concepts,  $\mathfrak{A}'$  is a model of  $\mathcal{T}$ , and so it is enough to show that  $b_0 \in (\neg C \sqcup D)^{\mathfrak{B}'}$  for all  $C \sqsubseteq D \in \mathcal{T}$ .

Let  $\mathfrak{B}'|_\ell$  be the structure obtained from  $\mathfrak{B}'$  by dropping all nodes of depth  $> \ell$ . We hook to every leaf  $b \in \mathfrak{B}'|_\ell$  of depth  $\ell$  a structure  $\mathfrak{B}_b$  so that  $\mathfrak{A}', X \preceq_{\text{horn}}^\ell \mathfrak{B}'', b_0$  for the resulting structure  $\mathfrak{B}''$ . As  $\mathcal{T}$  is preserved under global Horn simulations,  $\mathfrak{B}''$  is a model of  $\mathcal{T}$ . As  $\mathcal{T}$  has depth  $\leq \ell$ , we have  $b_0 \in (\neg C \sqcup D)^{\mathfrak{B}'}$  for all  $C \sqsubseteq D \in \mathcal{T}$ , as required. We come to the construction of the  $\mathfrak{B}_b$  for  $b$  a leaf of depth  $\ell$  in  $\mathfrak{B}'|_\ell$ . Since  $H^0, \dots, H^\ell$  is injective, there is a unique non-empty  $X_b \subseteq \text{dom}(\mathfrak{A}')$  such that  $(X_b, b) \in H^\ell$ . Observe that from  $(X_b, b) \in H^\ell$  it follows that  $X_b \subseteq A^{\mathfrak{A}'}$  iff  $b \in A^{\mathfrak{B}'}$  for any  $A \in \tau$ . Let  $\mathfrak{A}'_a$  be the tree-shaped substructure of  $\mathfrak{A}'$  rooted at  $a$ , for  $a \in X_b$ . Then we hook to  $b$  the structure  $\mathfrak{B}_b = \prod_{a \in X_b} \mathfrak{A}'_a$  by identifying  $(a \mid a \in X_b) \in \prod_{a \in X_b} \mathfrak{A}'_a$  with  $b$ . Using Lemma 2, it is readily checked that the resulting structure is as required.

In the finite model theory setting, we consider finite  $\mathfrak{A}$  and  $\mathfrak{B}$ . Then we can assume that the structures  $\mathfrak{B}'|_\ell$  and  $\mathfrak{A}'|_\ell$  are finite. Now, rather than hooking the (possibly infinite)  $\mathfrak{B}_b = \prod_{a \in X_b} \mathfrak{A}'_a$  to every leaf  $b$  of depth  $\ell$  in  $\mathfrak{B}'|_\ell$  we (i) replace all  $\mathfrak{A}'_a$  with  $a$  of depth  $\ell$  in  $\mathfrak{A}'$  by *finite* models  $\mathfrak{A}''_a$  of  $\mathcal{T}$

satisfying the same subconcepts of  $\mathcal{T}$  as  $\mathfrak{A}'_a$  in  $a$  and (ii) hook  $\prod_{a \in X_b} \mathfrak{A}''_a$  to  $b$ . Then  $\mathfrak{A}'', X \preceq_{\text{horn}}^\ell \mathfrak{B}''', b_0$  for the resulting finite structures  $\mathfrak{A}''$  and  $\mathfrak{B}'''$ , and  $\mathfrak{A}''$  is a model of  $\mathcal{T}$ .  $\square$

As there is only a finite number of *hornALC*-concepts and TBoxes of bounded depth in a finite vocabulary  $\tau$ , it follows from Theorems 6 and 7 that it is decidable whether an *ALC*-concept or TBox is equivalent to a *hornALC*-concept or TBox, respectively.

## VI. HORN GUARDED FRAGMENT *hornGF* OF FO

We extend *hornALC* to the Horn fragment, *hornGF*, of the guarded fragment of FO in the obvious way. *hornGF* contains numerous popular Horn DLs including those extending *hornALC* with inverse roles, the universal role, and role inclusions [4], [5]. We then generalize the Horn simulation games to guarded Horn simulation games for *hornGF* and prove an Ehrenfeucht-Fraïssé type definability theorem and a van Benthem style expressive completeness result for *hornGF*. Applications include an EXPTIME upper bound for model indistinguishability.

Let  $\tau$  be a vocabulary of predicate names  $R$  of arbitrary arity  $r_R \geq 0$ . The *guarded fragment*  $\text{GF}[\tau]$  of FO is defined by the following rules:

- $\text{GF}[\tau]$  contains the constants  $\top$  (truth) and  $\perp$  (falsehood);
- $\text{GF}[\tau]$  contains the *atomic formulas*  $R(\mathbf{x})$  and  $x = y$  with  $R \in \tau$ ;
- $\text{GF}[\tau]$  is closed under the connectives  $\wedge, \vee$ , and  $\neg$ ;
- if  $\varphi(\mathbf{xy})$  is in  $\text{GF}[\tau]$  with free variables among  $\mathbf{xy}$  and  $G(\mathbf{xy})$  is an atomic formula containing all the variables in  $\mathbf{xy}$ , then

$$\forall \mathbf{y} (G(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})), \quad \exists \mathbf{y} (G(\mathbf{xy}) \wedge \varphi(\mathbf{xy}))$$

are in  $\text{GF}[\tau]$  (these are called the universal and existential *guarded quantifiers* of  $\text{GF}[\tau]$ ).

If the particular vocabulary  $\tau$  is not relevant, we simply write  $\text{GF}$  for  $\text{GF}[\tau]$ . The *nesting depth of guarded quantifiers* in a formula  $\varphi$  in  $\text{GF}$ , or simply the *depth* of  $\varphi$ , is defined as the number of nestings of guarded quantifiers in  $\varphi$ . The formulas of the *positive existential guarded fragment*  $\text{GF}^\exists[\tau]$  of  $\text{GF}[\tau]$  are constructed from atomic formulas using  $\wedge, \vee$ , and the guarded existential quantifiers.

**Definition 5 (*hornGF*).** The fragment *hornGF* $[\tau]$  of  $\text{GF}[\tau]$  is given by the following grammar:

$$\begin{aligned} \varphi, \varphi' ::= & \perp \mid \top \mid x = y \mid R(\mathbf{x}) \mid \varphi \wedge \varphi' \mid \lambda \rightarrow \varphi \\ & \mid \exists \mathbf{y} (G(\mathbf{xy}) \wedge \varphi(\mathbf{xy})) \mid \forall \mathbf{y} (G(\mathbf{xy}) \rightarrow \varphi(\mathbf{xy})), \end{aligned}$$

where  $R \in \tau$ ,  $G(\mathbf{xy})$  are atomic formulas containing all the variables in  $\mathbf{xy}$ , and  $\lambda \in \text{GF}^\exists[\tau]$ .

*hornGF* is closely related to guarded tuple-generating dependencies (guarded tgds), a member of the Datalog $^\pm$  family of ontology languages for which query answering is in PTIME [43]. *Guarded tgds* are FO-formulas of the form  $\forall \mathbf{x} \forall \mathbf{y} (\psi(\mathbf{xy}) \rightarrow \exists \mathbf{z} \varphi(\mathbf{xz}))$  with conjunctions of atoms

$\psi(\mathbf{xy})$  and  $\varphi(\mathbf{xz})$  such that  $\psi$  contains an atom  $G(\mathbf{xy})$  guarding all the variables in  $\mathbf{xy}$ . Thus, in contrast to *hornGF*, guarded tgds have no quantifier alternation and can be regarded as a normal form for *hornGF*. In the full version, we give a polynomial time reduction of deduction and query answering in *hornGF* to the respective problems for guarded tgds by introducing fresh predicate names for complex formulas. We also provide a polynomial reduction in the converse direction. We note that satisfiability in *hornGF* has the same complexity as satisfiability in GF [39]: EXPTIME-complete if the arity of predicates is bounded and 2-EXPTIME-complete otherwise.

**Example 2.** The Horn formulas equivalent to the concept  $C_{\nabla}$  and TBox  $\mathcal{T}_{horn}$  from the proof of Theorem 1 are in *hornGF*. Thus, there are  $\mathcal{ALC}$ -concepts and TBoxes that are not equivalent to any *hornALC*-concepts or TBoxes, respectively, but nevertheless are equivalent to formulas in *hornGF*.

**Theorem 8.** (i) Every formula in *hornGF* is equivalent to a Horn formula.

(ii) There exists a sentence in GF—in fact, an  $\mathcal{ALC}$ -TBox—that is equivalent to a Horn sentence, but not equivalent to any *hornGF* sentence.

*Proof.* The proof of (i) is by a straightforward induction. For (ii), consider the TBox

$$\mathcal{T}_{guard} = \{ E \sqsubseteq \exists R. \top \sqcap \exists S. \top, E \sqcap \forall R. A \sqcap \forall S. B \sqsubseteq D \}.$$

It is equivalent to the Horn sentence

$$\forall x (E(x) \rightarrow \exists y_1 y_2 (R(x, y_1) \wedge S(x, y_2) \wedge ((A(y_1) \wedge B(y_2)) \rightarrow D(x))))$$

but, as shown in Example 4 below,  $\mathcal{T}_{guard}$  is not equivalent to any *hornGF*-sentence.  $\square$

Let  $\mathfrak{A} = (\text{dom}(\mathfrak{A}), (R^{\mathfrak{A}})_{R \in \tau})$  be a  $\tau$ -structure. Denote by  $\mathbf{a}$  a tuple  $a_1 \dots a_n$  of elements of  $\mathfrak{A}$  and set  $[\mathbf{a}] = \{a_1, \dots, a_n\}$ . A set  $X \subseteq \text{dom}(\mathfrak{A})$  is *guarded* in  $\mathfrak{A}$  if  $X$  is a singleton or  $R^{\mathfrak{A}}(\mathbf{a})$  for some  $R \in \tau$  and  $X = [\mathbf{a}]$ . A tuple  $\mathbf{a}$  is *guarded* if  $[\mathbf{a}]$  is guarded.

Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be structures. If  $\mathbf{a}$  and  $\mathbf{b}$  are tuples of the same length in  $\mathfrak{A}$  and  $\mathfrak{B}$ , respectively, we use  $p: \mathbf{a} \mapsto \mathbf{b}$  to denote the map from  $[\mathbf{a}]$  to  $[\mathbf{b}]$  with  $p(a_i) = b_i$ . If  $\mathbf{a}' = a_{i_1} \dots a_{i_k}$  is a subtuple of  $\mathbf{a}$ , then  $p(\mathbf{a}')$  denotes the subtuple  $p(a_{i_1}) \dots p(a_{i_k})$  of  $\mathbf{b}$ . The map  $p$  is a *homomorphism* from  $\mathfrak{A}|_{[\mathbf{a}]}$  to  $\mathfrak{B}|_{[\mathbf{b}]}$  if  $\mathbf{c} \in R^{\mathfrak{A}}$  implies  $p(\mathbf{c}) \in R^{\mathfrak{B}}$  for all  $R \in \tau$  and  $\mathbf{c}$  with  $[\mathbf{c}] \subseteq [\mathbf{a}]$ .

In this section, by a *pointed structure* we mean a pair  $\mathfrak{A}, X$  where  $X \subseteq \text{dom}(\mathfrak{A})$  is a nonempty set of *guarded tuples*, all of the same positive length. We again write  $\mathfrak{A}, \mathbf{a}$  for  $\mathfrak{A}, \{\mathbf{a}\}$ . We give the straightforward Ehrenfeucht-Fraïssé type characterization for  $\text{GF}^{\exists}$  needed for the characterization of *hornGF*. It is obtained from the standard guarded bisimulation characterization of GF [33] by replacing partial isomorphisms by homomorphisms and dropping the backward condition.

**Definition 6 (guarded simulation).** For  $\tau$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$ , a set  $Z$  of maps from guarded sets in  $\mathfrak{A}$  to guarded sets in

$\mathfrak{B}$  is called a *guarded simulation* if the following conditions hold for all  $p: \mathbf{a} \mapsto \mathbf{b}$  in  $Z$ :

(atom<sup>g</sup>)  $p: \mathfrak{A}|_{[\mathbf{a}]} \rightarrow \mathfrak{B}|_{[\mathbf{b}]}$  is a homomorphism;

(forth<sup>g</sup>) for every guarded tuple  $\mathbf{a}'$  in  $\mathfrak{A}$ , there exist a guarded tuple  $\mathbf{b}'$  in  $\mathfrak{B}$  and  $p'$  such that  $p': \mathbf{a}' \rightarrow \mathbf{b}'$  is in  $Z$  and  $p|_{[\mathbf{a}] \cap [\mathbf{a}']} = p'|_{[\mathbf{a}] \cap [\mathbf{a}]}$ .

We write  $\mathfrak{A}, \mathbf{a} \preceq_{gsim} \mathfrak{B}, \mathbf{b}$  if there exists a guarded simulation between  $\mathfrak{A}$  and  $\mathfrak{B}$  containing  $p: \mathbf{a} \mapsto \mathbf{b}$ .

In the same way as for  $\mathcal{ELU}$ , one can capture guarded simulations by *guarded simulation games* between two players such that player 2 has a winning strategy (can respond to any move of player 1) iff  $\mathfrak{A}, \mathbf{a} \preceq_{gsim} \mathfrak{B}, \mathbf{b}$ . We write  $\mathfrak{A}, \mathbf{a} \preceq_{gsim}^{\ell} \mathfrak{B}, \mathbf{b}$  if player 2 has a winning strategy for the guarded simulation game with  $\ell$  rounds.

We write  $\mathfrak{A}, \mathbf{a} \leq_{GF^{\exists}}^{(\ell)} \mathfrak{B}, \mathbf{b}$  if, for all formulas  $\lambda$  in  $\text{GF}^{\exists}$  (of depth  $\leq \ell$ ),  $\mathfrak{A} \models \lambda(\mathbf{a})$  implies  $\mathfrak{B} \models \lambda(\mathbf{b})$ .

**Theorem 9 (Ehrenfeucht-Fraïssé game for  $\text{GF}^{\exists}$ ).** For any finite vocabulary  $\tau$ , pointed  $\tau$ -structures  $\mathfrak{A}, \mathbf{a}$  and  $\mathfrak{B}, \mathbf{b}$ , and any  $\ell < \omega$ , we have<sup>2</sup>

$$\mathfrak{A}, \mathbf{a} \leq_{GF^{\exists}}^{\ell} \mathfrak{B}, \mathbf{b} \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq_{gsim}^{\ell} \mathfrak{B}, \mathbf{b}.$$

If  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite, then

$$\mathfrak{A}, \mathbf{a} \leq_{GF^{\exists}} \mathfrak{B}, \mathbf{b} \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq_{gsim} \mathfrak{B}, \mathbf{b}.$$

#### A. Simulations for *hornGF*

We introduce guarded Horn simulation games and prove an Ehrenfeucht-Fraïssé type definability result for *hornGF*. A *link* between structures  $\mathfrak{A}$  and  $\mathfrak{B}$  is a pair  $(P, \mathbf{b})$  with  $\mathbf{b}$  a guarded tuple in  $\mathfrak{B}$  and  $P$  a nonempty set of mappings  $p: \mathbf{b} \mapsto p(\mathbf{b})$  such that each  $p$  is a homomorphism from  $\mathfrak{B}|_{[\mathbf{b}]}$  to  $\mathfrak{A}|_{[p(\mathbf{b})]}$ . We denote by  $P[\mathbf{b}]$  the set  $\{p(\mathbf{b}) \mid p \in P\}$  and define the analogue of  $X R^{\uparrow} Y$  for guarded Horn simulations. If  $(P, \mathbf{b})$  is a link between  $\mathfrak{A}$  and  $\mathfrak{B}$ , and  $\mathfrak{A}, Y$  a pointed structure,  $R(\mathbf{x}_0 \mathbf{y})$  an atomic formula of the same arity as the tuples in  $Y$ , and  $\mathbf{b}_0$  a subtuple of  $\mathbf{b}$  of the same length as  $\mathbf{x}_0$ , then we say that  $Y$  is an  $R(\mathbf{b}_0 \mathbf{y})$ -successor of  $(P, \mathbf{b})$  when, for any  $p \in P$ , there exists a tuple  $\mathbf{a}$  with  $p(\mathbf{b}_0) \mathbf{a} \in Y$  and  $\mathfrak{A} \models R(p(\mathbf{b}_0) \mathbf{a})$ .

**Definition 7 (guarded Horn simulation).** A *guarded Horn simulation* between structures  $\mathfrak{A}$  and  $\mathfrak{B}$  is a set  $Z$  of links between  $\mathfrak{A}$  and  $\mathfrak{B}$  such that for all  $(P, \mathbf{b}) \in Z$ , we have:

(atom<sup>g</sup><sub>h</sub>) for all atomic formulas  $R(\mathbf{x})$  and tuples  $\mathbf{b}'$  with  $[\mathbf{b}'] \subseteq [\mathbf{b}]$ , if  $\mathfrak{A} \models R(p(\mathbf{b}'))$  for all  $p \in P$ , then  $\mathfrak{B} \models R(\mathbf{b}')$ ;

(forth<sup>g</sup><sub>h</sub>) for all sets  $Y$  of guarded tuples and atomic formulas  $R(\mathbf{x}_0 \mathbf{y})$ , if  $Y$  is an  $R(\mathbf{b}_0, \mathbf{y})$ -successor of  $(P, \mathbf{b})$ , then there exists  $(P', \mathbf{b}_0 \mathbf{b}') \in Z$  such that  $P'[\mathbf{b}_0 \mathbf{b}'] \subseteq Y$ ;

(back<sup>g</sup><sub>h</sub>) for every guarded tuple  $\mathbf{b}'$  in  $\mathfrak{B}$ , there exists a link  $(P', \mathbf{b}')$  in  $Z$  such that, for any  $p' \in P'$ , there exists  $p \in P$  with  $p|_{[\mathbf{b}] \cap [\mathbf{b}']} = p'|_{[\mathbf{b}] \cap [\mathbf{b}']}$ ;

<sup>2</sup>Here and in what follows the assumption that the tuples considered in Ehrenfeucht-Fraïssé characterizations are guarded is not essential. It is straightforward to modify the model comparison games in such a way that the characterizations hold for arbitrary tuples.

$(sim_h^g)$  there exists a guarded simulation between  $(\mathfrak{B}, \mathbf{b})$  and  $(\mathfrak{A}, p(\mathbf{b}))$  for every  $p \in P$ .

We write  $\mathfrak{A}, X \preceq_{ghsim} \mathfrak{B}, \mathbf{b}$  if there exists a guarded Horn simulation  $Z$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  such that  $X = P[\mathbf{b}]$  for some  $P$  with  $(P, \mathbf{b}) \in Z$ .

Lemma 1 linking Horn simulations with bisimulations can be lifted to the guarded case. In fact, any guarded bisimulation  $Z$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  defines a guarded Horn simulation  $Z' = \{(\{p^{-1}\}, \mathbf{b}) \mid p: \mathbf{a} \mapsto \mathbf{b} \in Z\}$  and if  $Z$  is a guarded Horn simulation with singleton  $P$  for every  $(P, \mathbf{b}) \in Z$ , then  $\{p^{-1} \mid (\{p\}, \mathbf{b}) \in Z\}$  is a guarded bisimulation (notice that, by  $(atom_h^g)$  and  $(sim_h^g)$ ,  $p$  is a partial isomorphism if  $P = \{p\}$ ).

On the other hand, as the moves of player 1 are no longer restricted to those along  $R^{\mathfrak{A}}$ ,  $R \in \tau$ , the relationship to products is subtler than in the Horn simulation case (Lemma 2).

**Example 3.** For  $i = 1, 2$ , let  $\mathfrak{A}_i = (\{a_i\}, A_1^{\mathfrak{A}_i}, A_2^{\mathfrak{A}_i})$ , where  $A_1^{\mathfrak{A}_1} = \{a_1\}$ ,  $A_2^{\mathfrak{A}_2} = \{a_2\}$ , and  $A_2^{\mathfrak{A}_1} = A_1^{\mathfrak{A}_2} = \emptyset$ . Then  $Z = \{(\{a_1, a_2\}, (a_1, a_2))\}$  is a Horn simulation between the disjoint union  $\mathfrak{A}$  of  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  and the product  $\mathfrak{A}_1 \times \mathfrak{A}_2$ , but it is not a guarded Horn simulation as  $Y = \{a_1\}$  is a  $(y = y)$ -successor of  $(\{a_1, a_2\}, (a_1, a_2))$  (with empty  $\mathbf{b}_0$ ) for which there is no link with  $Y$  as the first component in  $Z$ . Clearly, it is also not possible to expand  $Z$  to a guarded Horn simulation. We will revisit the relationship to products below.

For  $\ell < \omega$ , we define the relations  $\mathfrak{A}, X \preceq_{ghsim}^{\ell} \mathfrak{B}, \mathbf{b}$  in the obvious way following the definition Horn simulation games with  $\ell$  rounds. We write  $\mathfrak{A}, X \preceq_{hornGF}^{(\ell)} \mathfrak{B}, \mathbf{b}$  if for all formulas  $\varphi$  in *hornGF* (of depth  $\leq \ell$ ) the following holds: if  $\mathfrak{A} \models \varphi(\mathbf{a})$  for all  $\mathbf{a} \in X$ , then  $\mathfrak{B} \models \varphi(\mathbf{b})$ .

**Theorem 10 (Ehrenfeucht-Fraïssé game for *hornGF*).** For any finite vocabulary  $\tau$ , pointed  $\tau$ -structures  $\mathfrak{A}, \mathbf{a}$  and  $\mathfrak{B}, \mathbf{b}$ , and any  $\ell < \omega$ , we have

$$\mathfrak{A}, \mathbf{a} \preceq_{hornGF}^{\ell} \mathfrak{B}, \mathbf{b} \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq_{ghsim}^{\ell} \mathfrak{B}, \mathbf{b}.$$

If  $\mathfrak{A}$  and  $\mathfrak{B}$  are finite, then

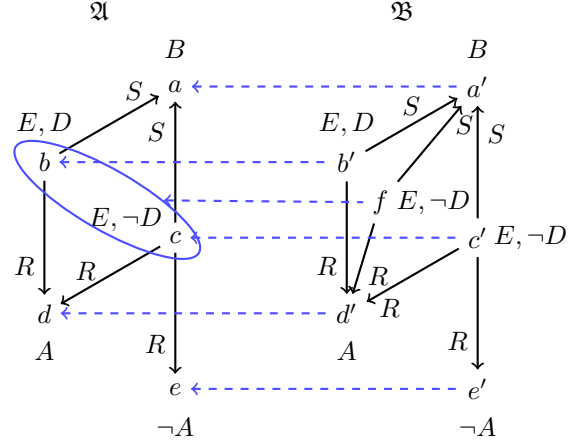
$$\mathfrak{A}, \mathbf{a} \preceq_{hornGF} \mathfrak{B}, \mathbf{b} \quad \text{iff} \quad \mathfrak{A}, \mathbf{a} \preceq_{ghsim} \mathfrak{B}, \mathbf{b}.$$

The proof of Theorem 10 is similar to that of Theorem 3 and given in the full version. In particular, one has to prove again a stronger version where the tuple  $\mathbf{a}$  is replaced by a set  $X$  of tuples. The existence of a winning strategy for player 2 in the guarded Horn simulation game is decidable in exponential time. Thus, it follows from Theorem 10 that entailment and equivalence in *hornGF* are decidable in EXPTIME.

**Theorem 11.** In *hornGF*, entailment, equivalence, and CBE are in EXPTIME. Moreover,  $\ell$ -entailment,  $\ell$ -equivalence, and  $\ell$ -CBE are in EXPTIME for binary encoding of  $\ell$  and in PSPACE for unary encoding.

In contrast to *hornALC*, it remains open whether the EXPTIME upper bound is tight. Using guarded Horn simulations, we now show that the TBox  $\mathcal{T}_{guard}$  from the proof of Theorem 8 is not equivalent to any *hornGF*-sentence.

**Example 4.** Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be the structures below.  $\mathfrak{B}$  is a copy of  $\mathfrak{A}$  with the extra node  $f$ ;  $\mathfrak{B}$  refutes  $\mathcal{T}_{guard}$  in  $f$ , but  $\mathfrak{A}$  is a model of  $\mathcal{T}_{guard}$ . A guarded Horn simulation  $Z$  between  $\mathfrak{A}$  and  $\mathfrak{B}$  is given by adding to the set of singleton links  $\{(\{u\}, u') \mid u \in \text{dom}(\mathfrak{A})\} \cup \{(\{uv\}, u'v') \mid (u, v) \in R^{\mathfrak{A}} \cup S^{\mathfrak{A}}\}$  the links  $(\{b, c\}, f)$ ,  $(\{bd, cd\}, fd')$ , and  $(\{ba, ca\}, fa')$ .



## B. Expressive Completeness for *hornGF*

Our next aim is to show that an FO-formula  $\varphi$  is equivalent to a *hornGF*-formula just in case it is preserved under guarded Horn simulations. This statement needs qualification, however, in two respects. First, our infinitary proof goes through only if we require the sets  $Y$  where player 1 moves to in condition  $(forth_h^g)$  to be intersections of FO-definable sets. Second, in contrast to *hornALC*, the language *hornGF* admits equality guards and is not local in the sense that the truth of a *hornGF*-formula  $\varphi(\mathbf{x})$  in  $\mathfrak{A}, \mathbf{a}$  is not determined by some neighbourhood of  $\mathbf{a}$  in the Gaifman graph of  $\mathfrak{A}$ . As a consequence, GF-sentences such as  $\varphi = \forall x A_1(x) \vee \forall x A_2(x)$  (with omitted equality guards) are not equivalent to any *hornGF*-sentence but preserved under guarded Horn simulations.

To deal with this issue, we lift the definition of guarded Horn simulations from single structures to families of structures. Let  $\mathfrak{A}_i, i \in I$ , be a family of disjoint structures. A set  $X$  of tuples in  $\bigcup_{i \in I} \text{dom}(\mathfrak{A}_i)$  intersects with all  $\mathfrak{A}_i, i \in I$ , if  $X$  contains at least one tuple from each  $\text{dom}(\mathfrak{A}_i)$ . For an open formula  $\varphi(\mathbf{x})$ , we write  $(\mathfrak{A}_i \mid i \in I) \models \varphi(\mathbf{a})$  if  $\mathbf{a}$  is a nonempty tuple in some  $\text{dom}(\mathfrak{A}_i)$  and  $\mathfrak{A}_i \models \varphi(\mathbf{a})$ . For closed  $\varphi$ , we write  $(\mathfrak{A}_i \mid i \in I) \models \varphi$  if  $\mathfrak{A}_i \models \varphi$  for all  $i \in I$ . A set  $X$  of tuples is *FO<sup>∞</sup>-definable* in  $(\mathfrak{A}_i \mid i \in I)$  if there is a set  $\Gamma(\mathbf{x})$  of FO-formulas with  $X = \{\mathbf{a} \mid \forall \varphi \in \Gamma(\mathbf{x}) (\mathfrak{A}_i \mid i \in I) \models \varphi(\mathbf{a})\}$ .

**Definition 8 (generalized guarded Horn simulation).** Let  $\mathfrak{A}_i, i \in I$ , be a family of disjoint structures,  $\mathfrak{A}$  the disjoint union of  $\mathfrak{A}_i, i \in I$ , and  $\mathfrak{B}$  a structure. A set  $Z$  of links between  $\mathfrak{A}$  and  $\mathfrak{B}$  is a *generalized guarded Horn simulation* between  $(\mathfrak{A}_i \mid i \in I)$  and  $\mathfrak{B}$  if all  $(P, \mathbf{b}) \in Z$  satisfy conditions  $(atom_h^g)$  and  $(back_h^g)$  from Definition 7 and

$(forth_h^{gg})$  for all sets  $Y$  of guarded tuples in  $\mathfrak{A}$  and atomic formulas  $R(\mathbf{x}_0\mathbf{y})$ , if  $Y$  is an  $R(\mathbf{b}_0\mathbf{y})$ -successor of  $(P, \mathbf{b})$  in  $\mathfrak{A}$  and

- $\mathbf{b}_0$  is not empty or
- $\mathbf{b}_0$  is empty and  $Y$  intersects with all  $\mathfrak{A}_i$ ,  $i \in I$ ,

then there is  $(P', \mathbf{b}_0 \mathbf{b}')$   $\in Z$  with  $P'[\mathbf{b}_0 \mathbf{b}'] \subseteq Y$ ;

( $\text{sim}_h^{gg}$ ) there exists a guarded simulation between  $(\mathfrak{B}, \mathbf{b})$  and  $(\mathfrak{A}_i, p(\mathbf{b}))$  for every  $p \in P$  and  $p(\mathbf{b})$  in  $\text{dom}(\mathfrak{A}_i)$ .

$Z$  is *FO-restricted* if ( $\text{forth}_h^{gg}$ ) holds for all  $\text{FO}^\infty$ -definable  $Y$ . We write  $(\mathfrak{A}_i \mid i \in I), X \preceq_{\text{ghorn}}^{\text{FO}} \mathfrak{B}, \mathbf{b}$  if there exists an FO-restricted generalized guarded Horn simulation  $Z$  between  $(\mathfrak{A}_i \mid i \in I)$  and  $\mathfrak{B}$  such that  $X = P[\mathbf{b}]$  for some  $(P, \mathbf{b}) \in Z$ .

Note that, as we modified ( $\text{forth}_h^{gg}$ ), the set  $Z$  in Example 3 is a generalized guarded Horn simulation. In fact, now Lemma 2 can be lifted to the guarded case: if  $\mathfrak{A}_i$ ,  $i \in I$ , is a family of structures, the set of all  $(P, f_1 \dots f_n)$  with  $f_1 \dots f_n$  a guarded tuple in  $\prod_{i \in I} \mathfrak{A}_i$  and  $p \in P$  just in case there exists  $i \in I$  such that  $p(f_j) = f_j(i)$ ,  $1 \leq j \leq n$ , is a generalized guarded Horn simulation between the disjoint union of the  $\mathfrak{A}_i$  and  $\prod_{i \in I} \mathfrak{A}_i$ .

A formula  $\varphi(\mathbf{x})$  is *preserved under FO-restricted generalized guarded Horn simulations* if  $(\mathfrak{A}_i \mid i \in I) \models \varphi(\mathbf{a})$  for all  $\mathbf{a} \in X$  and  $(\mathfrak{A}_i \mid i \in I), X \preceq_{\text{ghorn}}^{\text{FO}} \mathfrak{B}, \mathbf{b}$  imply  $\mathfrak{B} \models \varphi(\mathbf{b})$ .

**Theorem 12 (expressive completeness: *hornGF*).** *An FO-formula is equivalent to a *hornGF*-formula iff it is preserved under FO-restricted generalized guarded Horn simulations.*

*Proof.* (sketch) The implication  $(\Rightarrow)$  is straightforward. Conversely, suppose  $\varphi(\mathbf{x}_0)$  is preserved under FO-restricted generalized guarded Horn simulations. Let  $\text{cons}(\varphi)$  be the set of all  $\psi(\mathbf{x}_0)$  in *hornGF* entailed by  $\varphi(\mathbf{x}_0)$ . By compactness, it suffices to show  $\text{cons}(\varphi) \models \varphi$ . Let  $\mathfrak{B}$  be an  $\omega$ -saturated [52] model satisfying  $\text{cons}(\varphi)(\mathbf{b}_0)$  for some tuple  $\mathbf{b}_0$  in  $\text{dom}(\mathfrak{B})$ . We show  $\mathfrak{B} \models \varphi(\mathbf{b}_0)$ . For any tuple  $\mathbf{b}$  and tuple  $\mathbf{x}$  of variables of the same length as  $\mathbf{b}$ , we denote by  $\lambda_{\mathfrak{B}, \mathbf{b}}(\mathbf{x})$  the set of guarded existential positive  $\lambda(\mathbf{x})$  with  $\mathfrak{B} \models \lambda(\mathbf{b})$ . Let  $\mathcal{C}$  be the set of all sets  $\Gamma(\mathbf{x}_0)$  of FO-formulas with  $\mathfrak{B} \models \Gamma(\mathbf{b}_0)$  and such that  $\Gamma(\mathbf{x}_0) \cup \{\varphi(\mathbf{x}_0)\}$  is satisfiable and take, for any  $\Gamma(\mathbf{x}_0) \in \mathcal{C}$ , an  $\omega$ -saturated structure  $\mathfrak{A}_\Gamma$  and tuple  $\mathbf{a}_\Gamma$  with  $\mathfrak{A}_\Gamma \models (\Gamma \cup \{\varphi\})(\mathbf{a}_\Gamma)$ . Let  $\mathfrak{A}$  be the disjoint union of  $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$  and let  $Z$  be the set of pairs  $(X, \mathbf{b})$  such that

- (a) for any  $\psi(\mathbf{x}) \in \text{hornGF}$ , if  $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \psi(\mathbf{a})$  for all  $\mathbf{a} \in X$ , then  $\mathfrak{B} \models \psi(\mathbf{b})$ ;
- (b) there exists a set  $\Phi(\mathbf{x}) \supseteq \lambda_{\mathfrak{B}, \mathbf{b}}(\mathbf{x})$  of FO-formulas such that  $X$  is the set of all tuples  $\mathbf{a}$  with  $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C}) \models \Phi(\mathbf{a})$ .

Each  $(X, \mathbf{b}) \in Z$  can be regarded as a link  $(P, \mathbf{b})$  with  $X = P[\mathbf{b}]$ . As we work with  $\omega$ -saturated structures, one can show that  $Z$  is an FO-restricted generalized guarded Horn simulation between  $(\mathfrak{A}_\Gamma \mid \Gamma \in \mathcal{C})$  and  $\mathfrak{B}$ . Since  $\varphi(\mathbf{x}_0)$  is preserved under generalized guarded Horn simulations,  $\mathfrak{B} \models \varphi(\mathbf{b}_0)$ .  $\square$

## VII. CONCLUSION AND OUTLOOK

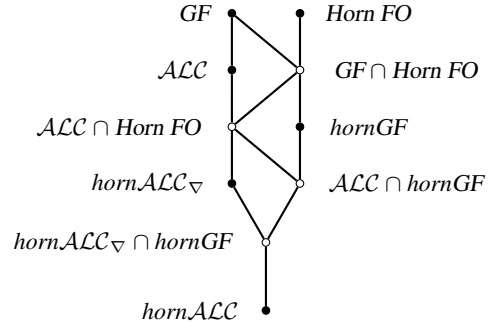
We have introduced model comparison games for *hornALC* and *hornGF* and obtained Ehrenfeucht-Fraïssé type definability and van Benthem style expressive completeness results. For *hornALC*, our results are ‘complete’: the characterizations hold in both classical and finite model theory settings without any restrictions on the players’ moves, and the straightforward EXPTIME upper bound for checking indistinguishability of

models and concept learnability using our model comparison games is tight. For more expressive *hornGF*, it remains open whether the characterization holds in the setting of finite model theory and whether the moves of the players have to be restricted to ‘saturated’ sets in the expressive completeness result. In this case, it is also open whether the EXPTIME upper bound for checking indistinguishability of models is tight.

A different line of open research problems arises from the fact that *hornALC* and *hornGF* do not capture the intersections of *ALC* (respectively, GF) and Horn FO. It is thus an open problem to find out whether there exists a ‘neat’ syntactic definition of the intersection of *ALC* and Horn FO such that, if an *ALC*-concept or TBox is equivalent to a Horn FO formula, then it is equivalent to a concept, or, respectively, TBox, satisfying this definition. The analogous question is also open for *hornGF* and Horn FO. The proofs of Theorems 1 and 8 suggest the following syntactic extension of *hornALC*.

**Example 5.** Denote by  $\mathcal{ELU}_\nabla$  the extension of  $\mathcal{ELU}$  with the  $\nabla$ -operator defined as  $\nabla R.C = \exists R.\top \sqcap \forall R.C$  and let *hornALC* $_\nabla$  be defined in the same way as *hornALC* (Definition 1) with the exception that now  $L$  is an  $\mathcal{ELU}_\nabla$ -concept. Then the concept  $C_\nabla$  from the proof of Theorem 1 (i) and the TBox  $\mathcal{T}_{\text{guard}}$  from the proof of Theorem 8 are clearly a *hornALC* $_\nabla$ -concept and TBox, respectively. So *hornALC* $_\nabla$  captures more from the intersection of *ALC* and Horn FO than *hornALC*. One can also show by an inductive argument that all *hornALC* $_\nabla$ -concepts and TBoxes are indeed equivalent to Horn FO formulas (details are in the full version). However, again this language does not fully capture the intersection in question as the TBox  $\mathcal{T}_{\text{horn}}$  from the proof of Theorem 1 (ii) is not equivalent to any *hornALC* $_\nabla$ -TBox. This can be shown by introducing model comparison games for *hornALC* $_\nabla$  (obtained by replacing the simulation game for  $\mathcal{ELU}$  with a game capturing  $\mathcal{ELU}_\nabla$ ) and showing that the Horn simulation from Example 1 (ii) preserves *hornALC* $_\nabla$ .

Taking into account the examples given in this paper, we arrive at the following lattice of languages and their intersections (modulo equivalence) where all inclusions are proper:



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## REFERENCES

- [1] F. Baader, D. Calvanese, D. McGuinness, D. Nardi, and P. Patel-Schneider, *The Description Logic Handbook: Theory, implementation and applications*. Cambridge University Press, 2003.
- [2] F. Baader, I. Horrocks, C. Lutz, and U. Sattler, *An Introduction to Description Logic*. Cambridge University Press, 2017.
- [3] M. Y. Vardi, “The complexity of relational query languages (extended abstract),” in *Proceedings of STOC*. ACM, 1982, pp. 137–146.
- [4] U. Hustadt, B. Motik, and U. Sattler, “Data complexity of reasoning in very expressive description logics,” in *Proceedings of IJCAI*, 2005, pp. 466–471.
- [5] —, “Reasoning in description logics by a reduction to disjunctive datalog,” *J. Autom. Reasoning*, vol. 39, no. 3, pp. 351–384, 2007.
- [6] H. Sturm, “Modal Horn classes,” *Studia Logica*, vol. 64, no. 3, pp. 301–313, 2000.
- [7] F. Baader, S. Brandt, and C. Lutz, “Pushing the  $\mathcal{EL}$  envelope,” in *Proceedings of IJCAI*. Professional Book Center, 2005, pp. 364–369.
- [8] D. Calvanese, G. D. Giacomo, D. Lembo, M. Lenzerini, and R. Rosati, “Tractable reasoning and efficient query answering in description logics: The DL-Lite family,” *J. Autom. Reasoning*, vol. 39, no. 3, pp. 385–429, 2007.
- [9] A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyashev, “The DL-Lite family and relations,” *J. Artif. Intell. Res. (JAIR)*, vol. 36, pp. 1–69, 2009.
- [10] T. Eiter, G. Gottlob, M. Ortiz, and M. Simkus, “Query answering in the description logic Horn- $\mathcal{SHIQ}$ ,” in *Proceedings of JELIA*, 2008, pp. 166–179.
- [11] M. Ortiz, S. Rudolph, and M. Simkus, “Query answering in the Horn fragments of the description logics  $\mathcal{SHOIQ}$  and  $\mathcal{SROIQ}$ ,” in *Proceedings of IJCAI*, 2011, pp. 1039–1044.
- [12] M. Bienvenu and M. Ortiz, “Ontology-mediated query answering with data-tractable description logics,” in *Proceedings of Reasoning Web*, 2015, pp. 218–307.
- [13] B. C. Grau, I. Horrocks, M. Krötzsch, C. Kupke, D. Magka, B. Motik, and Z. Wang, “Acyclicity conditions and their application to query answering in description logics,” in *Proceedings of KR*, 2012.
- [14] M. Bienvenu, P. Hansen, C. Lutz, and F. Wolter, “First order-rewritability and containment of conjunctive queries in Horn description logics,” in *Proceedings of IJCAI*, 2016, pp. 965–971.
- [15] B. Glimm, Y. Kazakov, and T. Tran, “Ontology materialization by abstraction refinement in Horn  $\mathcal{SHOIF}$ ,” in *Proceedings of AAAI*, 2017, pp. 1114–1120.
- [16] Y. Kazakov, “Consequence-driven reasoning for Horn- $\mathcal{SHIQ}$  ontologies,” in *Proceedings of IJCAI*, 2009, pp. 2040–2045.
- [17] M. Krötzsch, *Description Logic Rules*, ser. Studies on the Semantic Web. IOS Press, 2010, vol. 8.
- [18] M. Krötzsch, S. Rudolph, and P. Hitzler, “Complexities of Horn description logics,” *ACM Trans. Comput. Log.*, vol. 14, no. 1, pp. 2:1–2:36, 2013.
- [19] F. Baader, M. Bienvenu, C. Lutz, and F. Wolter, “Query and predicate emptiness in ontology-based data access,” *J. Artif. Intell. Res. (JAIR)*, vol. 56, pp. 1–59, 2016.
- [20] E. Botoeva, R. Kontchakov, V. Ryzhikov, F. Wolter, and M. Zakharyashev, “Games for query inseparability of description logic knowledge bases,” *Artif. Intell.*, vol. 234, pp. 78–119, 2016.
- [21] V. Gutiérrez-Basulto, J. C. Jung, and L. Sabellek, “Reverse engineering queries in ontology-enriched systems: The case of expressive Horn description logic ontologies,” in *Proceedings of IJCAI-ECAI-18*. AAAI Press, 2018.
- [22] E. Botoeva, C. Lutz, V. Ryzhikov, F. Wolter, and M. Zakharyashev, “Query inseparability for  $\mathcal{ALC}$  ontologies,” *Artif. Intell.*, 2019.
- [23] C. Lutz and F. Wolter, “The data complexity of description logic ontologies,” *Logical Methods in Computer Science*, vol. 13, no. 4, 2017.
- [24] A. Hernich, C. Lutz, F. Papacchini, and F. Wolter, “Horn-rewritability vs PTime query evaluation in ontology-mediated querying,” in *Proceedings of IJCAI-ECAI*. AAAI Press, 2018.
- [25] J. Lehmann, “DL-learner: Learning concepts in description logics,” *Journal of Machine Learning Research*, vol. 10, pp. 2639–2642, 2009.
- [26] L. Badea and S. Nienhuys-Cheng, “A refinement operator for description logics,” in *Proceedings of ILP*, 2000, pp. 40–59.
- [27] W. W. Cohen and H. Hirsh, “Learning the classic description logic: Theoretical and experimental results,” in *Proceedings KR*, 1994, pp. 121–133.
- [28] B. ten Cate and V. Dalmau, “The product homomorphism problem and applications,” in *Proceedings of ICDT*, 2015, pp. 161–176.
- [29] P. Barceló and M. Romero, “The complexity of reverse engineering problems for conjunctive queries,” in *Proceedings of ICDT*, 2017, pp. 7:1–7:17.
- [30] M. Arenas and G. I. Diaz, “The exact complexity of the first-order logic definability problem,” *ACM Trans. Database Syst.*, vol. 41, no. 2, pp. 13:1–13:14, 2016.
- [31] M. Grohe and M. Ritzert, “Learning first-order definable concepts over structures of small degree,” in *Proceedings of LICS 2017*, 2017, pp. 1–12.
- [32] V. Goranko and M. Otto, “Model theory of modal logic,” in *Handbook of Modal Logic*. Elsevier, 2007, pp. 249–329.
- [33] E. Grädel and M. Otto, “The freedoms of (guarded) bisimulation,” in *Johan van Benthem on Logic and Information Dynamics*, 2014, pp. 3–31.
- [34] E. Rosen, “Modal logic over finite structures,” *Journal of Logic, Language and Information*, vol. 6, no. 4, pp. 427–439, 1997.
- [35] M. Otto, “Modal and guarded characterisation theorems over finite transition systems,” *Ann. Pure Appl. Logic*, vol. 130, no. 1-3, pp. 173–205, 2004.
- [36] A. Dawar and M. Otto, “Modal characterisation theorems over special classes of frames,” *Ann. Pure Appl. Logic*, vol. 161, no. 1, pp. 1–42, 2009.
- [37] M. Otto, “Highly acyclic groups, hypergraph covers, and the guarded fragment,” *J. ACM*, vol. 59, no. 1, pp. 5:1–5:40, 2012.
- [38] H. Andréka, I. Németi, and J. van Benthem, “Modal languages and bounded fragments of predicate logic,” *J. Philosophical Logic*, vol. 27, no. 3, pp. 217–274, 1998.
- [39] E. Grädel, “On the restraining power of guards,” *J. Symb. Log.*, vol. 64, no. 4, pp. 1719–1742, 1999.
- [40] V. Bárány, G. Gottlob, and M. Otto, “Querying the guarded fragment,” *Logical Methods in Computer Science*, vol. 10, no. 2, 2014.
- [41] A. Hernich, C. Lutz, F. Papacchini, and F. Wolter, “Dichotomies in ontology-mediated querying with the guarded fragment,” in *Proceedings of PODS*, 2017.
- [42] A. Cali, G. Gottlob, and M. Kifer, “Taming the infinite chase: Query answering under expressive relational constraints,” *J. Artif. Intell. Res. (JAIR)*, vol. 48, pp. 115–174, 2013.
- [43] A. Cali, G. Gottlob, and T. Lukasiewicz, “A general datalog-based framework for tractable query answering over ontologies,” in *Proceedings of PODS*, 2009, pp. 77–86.
- [44] N. Kurtonina and M. de Rijke, “Expressiveness of concept expressions in first-order description logics,” *Artif. Intell.*, vol. 107, no. 2, pp. 303–333, 1999.
- [45] C. Lutz, R. Piro, and F. Wolter, “Description logic TBoxes: Model-theoretic characterizations and rewritability,” in *IJCAI*, 2011.
- [46] V. Bárány, B. ten Cate, and L. Segoufin, “Guarded negation,” *J. ACM*, vol. 62, no. 3, pp. 22:1–22:26, 2015.
- [47] D. Gorín and L. Schröder, “Simulations and bisimulations for coalgebraic modal logics,” in *Proceedings of CALCO 2013*, 2013, pp. 253–266.
- [48] L. Schröder, D. Pattinson, and T. Litak, “A van Benthem/Rosen theorem for coalgebraic predicate logic,” *J. Log. Comput.*, vol. 27, no. 3, pp. 749–773, 2017.
- [49] P. Wild, L. Schröder, D. Pattinson, and B. König, “A van Benthem theorem for fuzzy modal logic,” in *Proceedings of LICS*, 2018, pp. 909–918.
- [50] J. C. C. McKinsey, “The decision problem for some classes of sentences without quantifiers,” *J. Symb. Log.*, vol. 8, no. 2, pp. 61–76, 1943.
- [51] A. Horn, “On sentences which are true of direct unions of algebras,” *J. Symb. Log.*, vol. 16, no. 1, pp. 14–21, 1951.
- [52] C. C. Chang and H. J. Keisler, *Model Theory*. Elsevier, 1990.
- [53] W. Hodges, *Model Theory*. Cambridge University Press, 1993.
- [54] J. Weinstein, “First order properties preserved by direct product,” Ph.D. dissertation, Univ. Wisconsin, Madison, 1965.
- [55] L. A. Nguyen, “On the complexity of fragments of modal logics,” in *Proceedings of AiML*. King’s College Publications, 2004, pp. 249–268.
- [56] L. Fariñas del Cerro and M. Penttonen, “A note of the complexity of the satisfiability of modal Horn clauses,” *J. Log. Program.*, vol. 4, no. 1, pp. 1–10, 1987.
- [57] C. C. Chen and I. P. Lin, “The computational complexity of satisfiability of temporal Horn formulas in propositional linear-time temporal logic,” *Information Processing Letters*, vol. 45, no. 3, pp. 131–136, 1993.

- [58] D. Bresolin, E. Muñoz-Velasco, and G. Sciavicco, “On the expressive power of sub-propositional fragments of modal logic,” in *Proceedings of GandALF*, 2016, pp. 91–104.
- [59] C. Lutz and F. Wolter, “Deciding inseparability and conservative extensions in the description logic  $\mathcal{EL}$ ,” *Journal of Symbolic Computation*, vol. 45, no. 2, pp. 194–228, 2010.
- [60] D. Harel, O. Kupferman, and M. Y. Vardi, “On the complexity of verifying concurrent transition systems,” *Inf. Comput.*, vol. 173, no. 2, pp. 143–161, 2002.